A STUDY OF CERTAIN FIN EQUATIONS USING LIE SYMMETRY ANALYSIS

BY

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I dedicate my Dissertation work to my family. A special feeling of gratitude to my loving parents, my wife, my son, my daughters, my brothers, my sisters and my uncles.

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DISSERTATION ABSTRACT

NAME: SAEED MOHAMMED SALMAN ALI

TITLE OF STUDY: **A Study of Certain Fin Equations Using Lie Symmetry Analysis**

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A type of evolution equation arising in industrial applications has been studied from Lie symmetry point of view. We consider the nonlinear (2+1)-dimensional fin equation in cylindrical and spherical coordinates. A complete classification of thermal conductivity and heat transfer coefficient is obtained. Reductions via two dimensional Lie subalgebra to ordinary differential equations are performed. Exact solutions for some interesting cases are found. Multiplier approach to establish conserved vectors for nonlinear (2+1)-dimensional fin equation in cylindrical coordinates is used. Also, double reduction for nonlinear (2+1)-dimensional fin equation in cylindrical coordinates is performed. Exact solutio, for a particular case of thermal conductivity and heat transfer, is found.

ملخص بحث
درجة الدكتوراة في الفلسفة

الاســــــــــم: **سعيد محمد سلمان علي**

عنوان الرسالة: دراسة بعض معادلات الزعنفة باستخدام التحليل التناظري ل **لي.**

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في هذه الرسالة قمنا بدراسة فئه من معادلات التطور التي تظهر في التطبيقات الصناعية بواسطة التحليل التماثلي. لقد افترضنا معادلة الزعنفه الغير خطية ذات (2+1) بعد في الاحداثيات الاسطوانية والكروية. حيث قمنا بتصنيفا تماثليا كاملا للموصلية الحرارية ومعامل انتقال الحرارة. بعد ذلك استخدمنا محاور التماثل لتخفيض هذه المعادلة الى معادلات تفاضلية اعتيادية باستخدام جبريات جزئية ذات بعدين ومن ثم ايجاد بعض الحلول التامة التحليلة لبعض الحالات. ايضا استخدمنا طريقة المضاعف لانشاء متجهات الحفظ لمعادلة الزعنفة الغير خطية ذات (2+1) بعد في الاحداثيات الاسطوانية. بعد ذلك اجرينا التخفيض المزدوج لهذه المعادلة وحصلنا على ضؤ العلاقة بين محاور التماثل وقوانين الحفظ للمعادلة التفاضلية الجزئية على حل تام في حالة خاصة.

Introduction

Most Physical phenomena are modeled in terms of partial differential equations (PDEs). There have been extensive studies in the theory of differential equations (DEs) and more extensively linear differential equations. However, due to complexities of the models, non-linearities arise. The non-linear differential equations are more challenging in that very few methods are available that apply to a wide class of such problems. Perturbation and variational methods besides a variety of numerical techniques are often employed, e.g., [22, 14]. Some recent approaches have been based upon numerical approximations, perturbation technique and decomposition expansion methods, e.g., [72, 48]. In last few decades, attention of researchers have been made to use Lie symmetry analysis [5, 6, 8, 12]. The idea of this method was given by Sophus Lie in the latter part of the nineteenth century [40, 39]. He introduced the notion of Lie point symmetry that can be used to reduce the number of independent variables or the order of DE.

Roughly, a symmetry is a change or transformation that leaves an object apparently unchanged. For instance any rotation of a circle about its centre is a symmetry. Different objects can have different degrees of symmetries: intuitively a circle has more symmetries than a rectangle. Therefore symmetry can be employed as a classification criterion. In the case of DEs a symmetry is an invertible transformation of the dependent and independent variables that does not affect the form of the underlying equation or system of equations. Looking at a DE one can deduce symmetries like translations, scalings and rotations. Certain discrete transformations can also be deduced by inspection. In general finding all the symmetries is a difficult task and requires an algorithmic approach. If we consider symmetries that depend continuously upon a one-parameter and that constitute a group, we can use Lie's algorithm to compute them. In this algorithm the symmetry conditions (also known as the determining equations) upon expansion yield an over-determined system of linear homogeneous PDEs which are solved for symmetries. Most problems even of no interest generating the determining equations turns out to be tedious. Fortunately Lie's method for calculating symmetries is algorithmic and can be implemented using packages for symbolic computation e.g., (Mathematica, Maple, Reduce, MuPad). In local terms the Lie group is a group of transformations which are defined in the neighborhood of an identity transformation. In this context the Lie groups are understood by the vector fields (and their algebras) that are generally known as infinitesimal symmetry generators. The original Lie group theory was extended to a global theory in the 1930s by Elie Cartan and Herman Weyl [62]. Emmy Noether later used the Lie symmetries to prove a relationship between symmetries of a variational problem and the conservation laws of the Euler-Lagrange equations [55]. Thereafter this area of study has been attracting many of researchers [44] even from other disciplines in which DEs play an important role. Nowadays modern group analysis brings together many researchers in various fields of natural sciences and engineering. Lie symmetry theory plays a vital role in fluid or solid mechanics, modern physics, biological and physical systems, stochastic forecasting in finance and meteorology and across all areas in engineering technology.

Lie's approach has been used by a number of researchers in the past thirty years, e.g., [30, 44, 45, 73]. Moreover, a good reference book that contains symmetries was authored by Ibragimov [28]. In spite of efforts to popularize Lie's work through publications, the application of Lie group theory to DEs remained dormant until Ovsiannikov [59] revived it in the late 1950s. Thereafter, the Lie group theory has been applied in many problems (described by linear or nonlinear equations) modeling some physical or abstract phenomena. The basic theory and developments in the Lie group analysis of DEs can be obtained from [7, 8, 26, 27, 56, 58] and the literature that followed, some of which are unpublished lecture notes and theses [15, 41, 59, 71]. The mathematical formulation of symmetries was already present in the theory of algebraic equations as developed by Galois [69]. In fact he introduced and established the concept of group (see Stubhaug [67], p.114). Lie discovered that the concept of symmetry could be used to obtain solutions of DEs. He considered only the local symmetries (point and contact symmetries). Point symmetries depend only upon independent and dependent variables, whereas contact symmetries can also depend upon the first-order derivatives of the dependent variables. From around 1970s until today there has been an interest in exploring nonlocal symmetries (see [3, 1, 7, 13, 54, 46, 42] just a few to mention). These are the symmetries that depend on integrals of the dependent variables. Ever since the discovery of the application of Lie group theory to DEs, the researchers developed the work of Lie in many directions. These include the following: Generation of new solutions from known ones [68], linearization of ordinary differential equations (ODEs) and PDEs [25], construction of equivalence group, solving group classification problems [63], reductions of PDEs (by invariant or similarity solutions) [20], construction of generalized local symmetries and nonlocal symmetries [74], solving initial and boundary value problems [38], approximate symmetries [35], symmetries of stochastic DEs [17], symmetries of integro differential equations, symmetries of difference equations [47], symmetries of functional differential equations, symmetries of geodesic equations [16], construction of conservation laws [2], construction of invariants of algebraic equations and DEs [57], etc.

In this work, we use this approach to analyze one type of evolution equation that arises in industrial applications. We consider the so-called fins which are widely used in industry as heat exchange surfaces. The fins are surfaces that increase the rate of heat transfer to or from surrounding environment by increasing convection. Such a heat exchange through fins occurs in industrial applications such as compressors, air conditioners and air craft engines. These fins are of different shapes and are described by variety of mathematical models [36]. It was perhaps Harper and Brown [21] who formulated mathematical model of the fin. Subsequently, Garden [18] assuming that the thermal conductivity of the fin is constant and there are no source or sinks present, derived the fin equation in the following form:

$$\frac{d}{dx}(w\frac{du}{dx}) - hu = 0 \tag{1}$$

where w is the base thickness of fin, h heat transfer coefficient and u is the temperature. More recently, Pakdemirli and Sahin [60, 61] have studied the problem in which the thermal conductivity is assumed to be temperature dependent as follows:

$$\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) - N^2 f(x)u = u_t, \tag{2}$$

where $k(u)$ is the thermal conductivity, N the fin constant and $f(x)$ heat transfer coefficient. They used the Lie symmetries of the governing PDE Eq.(2). Bokhari, Kara, & Zaman [11] further studied the above nonlinear fin equation Eq. (2)). They considered group theoretic analysis that led to some new exact solutions. Vaneeva *et.al.* [70] performed a Lie group classification of the nonlinear (1+1) dimensional fin equation and presented exact solutions. Moitshekia, Hayat & Malik [52] obtained some exact solutions of the Eq.(2) by considering a power law form of the thermal conductivity. Moitsheki [49] also considered a radial one-dimensional fin to steady heat transfer by assuming the fin equation in the following form

$$\frac{A_p}{r}\frac{d}{dr}\left(rf(r)k(u)\frac{du}{dr}\right) = h(u)\left(u - u_a\right), \quad r_b < r < r_a, \tag{3}$$

where k and h are the non-uniform thermal conductivity and heat transfer coefficients, respectively, depending on the temperature. Some exact solutions are obtained for thermal diffusion for fin with a rectangular profile and hyperbolic profile. Continuing their investigation, Moitsheki [49] also studied a steady heat transfer problem of a longitudinal fin with triangular and parabolic profiles by considering the following problem:

$$A_p \frac{d}{dx}\left(F(x)k(u)\frac{du}{dx}\right) = h(u)\left(u-u_a\right), \qquad 0 < x < L. \tag{4}$$

where A_p represents the profile area, $F(x)$ fin profile, and k and h respectively represent non-uniform thermal conductivity and heat transfer coefficient depending on the temperature.

Considering a state heat transfer problem for a rectangular fin, Moitsheki and Rowjee [50] discussed the (1+1) dimensional problem

$$\frac{\partial}{\partial y_1}\left(k(u)\frac{\partial u}{\partial y_1}\right) + \frac{\partial}{\partial x_1}\left(k(u)\frac{\partial u}{\partial x_1}\right) = s(u). \tag{5}$$

in which, u is the dimensionless temperature, x_1 and y_1 are longitudinal and transverse coordinates, s internal heat generation function, and k thermal conductivity. Employing the Lie symmetry analysis to classify the internal heat generating function, they obtained some reductions of the fin equation as well as exact solutions. In the same series of papers Moitsheki and Harley [51] considered a two-dimensional pin shape fin equation with length L and radius R and having the form.

$$\frac{1}{R}\frac{\partial}{\partial R}\left(Rk(u)\frac{\partial u}{\partial R}\right) + \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) = s(u). \tag{6}$$

Using Lie symmetry approach, they obtained certain solutions of this equation for different cases of $s(u)$.

The studies mentioned above are mostly restricted to (1+1) dimensions.

In this dissertation, we extend the (1+1) nonlinear fin equation to (2+1) by considering the governing equation

$$\mathbf{div}(k(u)\mathbf{grad}\, u) - N^2 f(x)u = u_t. \tag{7}$$

Then we study the above equation in terms of cylindrical and spherical coordinates. That is, in case of cylindrical coordinates, the above equation has the form

$$\frac{1}{r}\frac{\partial}{\partial r}(rk(u)\frac{\partial u}{\partial r}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\frac{1}{r}k(u)\frac{\partial u}{\partial \theta}) + \frac{\partial}{\partial z}(k(u)\frac{\partial u}{\partial z}) - N^2 f(r)u = u_t. \tag{8}$$

Ignoring the term $\frac{\partial}{\partial z}(k(u)\frac{\partial u}{\partial z})$, and replacing the radial coordinates r by x and angle θ by y, then the cylindrical coordinates of (2+1) fin equation becomes

$$\frac{1}{x}\frac{\partial}{\partial x}\left(xk(u)u_x\right) + \frac{1}{x}\frac{\partial}{\partial y}\left(\frac{1}{x}k(u)u_y\right) - N^2 f(x)u = u_t. \tag{9}$$

Similarly, the Eq.(7) transforms in spherical coordinates to the following form

$$\begin{aligned}&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k(u)\frac{\partial u}{\partial r}\right) + \frac{1}{rsin\theta}\left[\frac{\partial}{\partial \theta}\left(\frac{1}{r}sin\theta k(u)\frac{\partial u}{\partial \theta}\right)\right.\\ &\left.+\frac{\partial}{\partial \phi}\left(\frac{1}{rsin\theta}k(u)\frac{\partial u}{\partial \phi}\right)\right] - N^2 f(x)u = u_t.\end{aligned} \tag{10}$$

Ignoring the term $\frac{\partial}{\partial \phi}(\frac{1}{rsin\theta}k(u)\frac{\partial u}{\partial \phi})$, under the assumption of azimuthal symmetry, and replacing the radial coordinates r by x and angle θ by y, the spherical coordinates of (2+1) fin equation becomes

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2k(u)\frac{\partial u}{\partial x}\right)+\frac{1}{x\sin(y)}\frac{\partial}{\partial y}\left(\frac{1}{x}\sin(y)k(u)\frac{\partial u}{\partial y}\right)-N^2f(x)u=u_t. \tag{11}$$

A complete classification of $k(u)$ and $f(x)$ in both cases is obtained by using Lie symmetry analysis. Also, using two dimensional subalgebra, we carry out reductions to ODEs and whenever possible, obtain exact solutions.

Obtaining the conservation laws associate with symmetries is an important subject, and many authors have considered it (see e.g. [2, 4, 37, 53, 64]). We use the multiplier method to establish conservation laws that do not depend upon the Euler-Lagrange formulation of the DEs (see,[2]). This method is based on the well known result that the Euler-Lagrange operator annihilate a total divergence (see[23]). We use this method to construct conservation laws of particular case of cylindrical (2+1) fin equation and then use one of the Lie-symmetries of (9) that is associated with the resulting conservation law to obtain the double reduction (see, [9, 33, 53]). The double reduction of PDE occurs in two steps, first step is to reduce the independent variables and second step is to reduce the order of ODE (see [64, 65]).

Organization of this dissertation is as follows:

In the first chapter, we provide the basic concepts to tackle our investigations. We introduce the definitions, theory of continuous groups to find the Lie symmetries, multiplier method to find conservation laws and the multiplier method to achieve double reduction.

In the second chapter, we present a complete classification of nonlinear fin equation in cylindrical coordinates by using Lie symmetry analysis. Also, we carry out reductions and obtain possible exact solutions in some cases.

In the third chapter, we use the technique of invariance and multipliers to construct the conservation laws of cylindrical fin equation. We use the resulting conservation laws with associated symmetries of fin equation in cylindrical form to carry out the double reduction.

The fourth chapter provides a complete classification of fin equation in spherical coordinates using Lie symmetry analysis. Some reductions and exact solutions are obtained.

Some results and recommendations for future work are drawn in the fifth chapter.

CHAPTER 1

Basic Concepts

1.1 Introduction

In this chapter, we introduce the basic concepts that are needed to study the non-linear fin equation appearing in industrial applications. We provide the concepts of Lie Symmetry method, multiplier approach to construct conservation laws, and double reduction.

1.2 Lie Groups

The main goal of the theory of Lie groups is to describe the symmetry of analytical structures, especially in mathematical analysis, physics and geometry. So that, a Lie group lie at the intersection of algebra and geometry.

1.2.1 Groups

Definition 1.1 [5] A group G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:

(i) Closure property. For any elements a and b of G, $\phi(a, b)$ is an element of G,

(ii) Associative property, For any elements a, b, c of G:

$$\phi(a, \phi(b, c)) = \phi(\phi(a, b), c),$$

(iii) Identity element. There exists a unique identity element$e$ of G such that for any element a of G:

$$\phi(a, e) = \phi(e, a) = a$$

(iv) inverse element. For any element a of G there exists a unique inverse element a^{-1} in G such that

$$\phi(a, a^{-1}) = \phi(a^{-1}, a) = e$$

Definition 1.2 A group G is Abelian if $\phi(a, b) = \phi(b, a)$ holds for all elements a and b in G.

Definition 1.3 A subset H of G is said to be a subgroup of G if it satisfies the same axioms of group G with the same law of composition ϕ.

Example 1.1 The following are examples of groups:

(1) $\mathbb{R}^n$ with the a law of composition $\phi(a, b) = a + b,\ \ a, b \in \mathbb{R}^n$,

(2) $\mathbb{R} \setminus \{0\}$ with the a law of composition $\phi(a, b) = a \times b,\ \ a, b \in \mathbb{R} \setminus \{0\}$,

(3) $\mathbb{R}^+$ with the a law of composition $\phi(a, b) = a \times b,\ \ a, b \in \mathbb{R}^+$.

1.3 Group of Transformation

Definition 1.4 Let Λ be a set of transformations defined by

$$\Lambda(x;\varepsilon) = x^*$$

where $x \in D \subset \mathbb{R}^n$ and the parameter $\varepsilon \in S \subset \mathbb{R}$, with a law of composition $\phi(\varepsilon, \delta)$ for all $\varepsilon, \delta \in S$, then we say Λ one- parameter group of transformation if it satisfies the following conditions:

(i) Λ are one-to-one onto D for each $\varepsilon \in S$.

(ii) (S, ϕ) is a group.

(iii) For each $x \in D$, $x^* = x$ when $\varepsilon = \varepsilon_0$ corresponds to the identity e of the group (S, ϕ), i.e., $\Lambda(x;\varepsilon_0) = x$,

(iv) If $x^* = \Lambda(x;\varepsilon)$, $x^{**} = \Lambda(x^*;\delta)$, then $x^{**} = \Lambda(x;\phi(\varepsilon,\delta))$.

1.4 Lie Group of Transformation

Definition 1.5 We say that the one-parameter group of transformation $\Lambda(x;\varepsilon) = x^*$ with composition law ϕ, is a Lie group of transformation of one-parameter if:

(i) The parameter ε is continuous. i.e., the set S is an interval in $\mathbb{R}$,

(ii) Λ is infinitely differentiable with respect to x in D and an analytic function of ε in S.

(iii) The composition function $\phi(\varepsilon;\delta)$ is an analytic function.

Example 1.2 A group of translation in the plane given by

$$\tilde{x} = x + \varepsilon, \quad and \quad \tilde{y} = y + \varepsilon.$$

with composition law $\phi(\varepsilon, \delta) = \varepsilon + \delta$ and identity $\varepsilon_0 = 0$, forms Lie group.

Example 1.3 The group of scaling in the plane given by

$$\tilde{x} = e^{\varepsilon} x, \quad and \quad \tilde{y} = e^{\varepsilon} y.$$

with law of composition $\phi(\varepsilon, \delta) = \varepsilon + \delta$ and identity element $\varepsilon_0 = 0$, represents Lie group of transformation.

Example 1.4 The group of rotation transformation given by

$$\tilde{x} = x\cos\varepsilon - y\sin\varepsilon, \qquad \tilde{y} = x\sin\varepsilon + y\cos\varepsilon,$$

with law of composition $\phi(\varepsilon, \delta) = \varepsilon + \delta$ and $\varepsilon_0 = 0$, forms Lie group of transformation.

Example 1.5 The transformation given by

$$\tilde{x} = -x, \quad and \quad \tilde{y} = -y.$$

since,

$$\tilde{\tilde{x}} = x, \quad \tilde{\tilde{y}} = y.$$

Therefore, this transformation does not form a Lie group.

1.5 Infinitesimal Transformation

Let Λ^* be a one parameter Lie group of transformation given by

$$\Lambda^* = \Lambda(x;\varepsilon) \tag{1.1}$$

with the identity $\varepsilon = 0$ and composition law ϕ. Expand (1.1) about $\varepsilon = 0$, to get

$$x^* = x + \varepsilon \left(\frac{\partial \Lambda(x;\varepsilon)}{\partial \varepsilon}\bigg|_{\varepsilon=0} \right) + O(\varepsilon^2). \tag{1.2}$$

Defining $\xi(x) = \frac{\partial \Lambda(x;\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}$, then the transformation $x + \varepsilon\xi(x)$ is called the infinitesimal transformation of the Lie group of transformation (1.1) and $\xi(x)$ are called the infinitesimals of (1.1).

Example 1.6 Let $\hat{\Lambda}$ be a Lie group of transformation defined by

$$\hat{\Lambda}(x,y) = (\hat{x},\hat{y}), \quad (x,y) \in \mathbb{R}^2 \tag{1.3}$$

such that

$$\hat{x} = x + \varepsilon \left.\frac{\partial \hat{x}}{\partial \varepsilon}\right|_{\varepsilon=0} + \ldots,$$
$$\hat{y} = y + \varepsilon \left.\frac{\partial \hat{y}}{\partial \varepsilon}\right|_{\varepsilon=0} + \ldots. \tag{1.4}$$

Using $\left.\frac{\partial \hat{x}}{\partial \varepsilon}\right|_{\varepsilon=0} = \xi(x,y)$ and $\left.\frac{\partial \hat{y}}{\partial \varepsilon}\right|_{\varepsilon=0} = \eta(x,y)$ into (1.4), we get

$$\hat{x} = x + \varepsilon\xi(x,y) + \ldots,$$
$$\hat{y} = y + \varepsilon\eta(x,y) + \ldots. \tag{1.5}$$

The (1.5) are called the infinitesimal transformation of (1.3).

Theorem 1.1 (**First Fundamental Theorem of Lie [5]**) There exists a parameterizations $\tau(\varepsilon)$ such that the Lie group of transformation (1.1) is equivalent to the solution of an initial value problem for a system of first-order ODEs given by

$$\frac{dx^*}{d\tau} = \xi(x^*), \tag{1.6}$$

with $x^* = x \quad when \quad \tau = 0.$

1.5.1 Infinitesimal Generator

Definition 1.6 The infinitesimal generator of one-parameter Lie group transformation (1.1) is defined by

$$X = X(x) = \xi(x) \cdot \nabla = \sum_{k=1}^{n} \xi_k(x) \frac{\partial}{\partial x_k}, \tag{1.7}$$

where $\xi_k = \frac{\partial x_k^*}{\partial \varepsilon}\Big|_{\varepsilon=0}$ and $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$.

Theorem 1.2 [5] The one parameter Lie group of transformations $\Lambda^* = \Lambda(x;\varepsilon)$ is equivalent to

$$x^* = e^{\varepsilon X} x = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} X^k x, \tag{1.8}$$

where the operator X is defined by (1.7).

Example 1.7 Consider the rotation group given by

$$\begin{aligned} \tilde{x} &= x\cos\varepsilon + y\sin\varepsilon, \\ \tilde{y} &= -x\sin\varepsilon + y\cos\varepsilon. \end{aligned} \tag{1.9}$$

The infinitesimal for (1.9) is given by

$$\xi(x,y) = \frac{\partial \tilde{x}}{\partial \varepsilon}\bigg|_{\varepsilon=0} = y, \quad \textit{and} \quad \eta(x,y) = \frac{\partial \tilde{y}}{\partial \varepsilon}\bigg|_{\varepsilon=0} = -x.$$

which define the infinitesimal symmetry generator as

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

In accordance with theorem 1.2 the Lie series to above infinitesimal generator can be obtained as follows:

$$(\tilde{x}, \tilde{y}) = (e^{\varepsilon X} x, e^{\varepsilon X} y),$$

Thus,

$$\begin{aligned}
\bar{x} = e^{\varepsilon X}x &= \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}X^k x \\
&= X^0x + \frac{\varepsilon^1}{1!}X^1x + \frac{\varepsilon^2}{2!}X^2x + \frac{\varepsilon^3}{3!}X^3x + \dots \\
&= x - \frac{\varepsilon^1}{1!}y - \frac{\varepsilon^2}{2!}x + \frac{\varepsilon^3}{3!}y + \dots \\
&= \left(1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} + \dots\right)x + \left(\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} + \dots\right)y \\
&= x\cos\varepsilon + y\sin\varepsilon
\end{aligned}$$

Similarly,

$$\begin{aligned}
\bar{y} = e^{\varepsilon X}y &= \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}X^k y \\
&= X^0y + \frac{\varepsilon^1}{1!}X^1y + \frac{\varepsilon^2}{2!}X^2y + \frac{\varepsilon^3}{3!}X^3y + \dots \\
&= \left(1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} + \dots\right)y - \left(\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} + \dots\right)x \\
&= y\cos\varepsilon - x\sin\varepsilon.
\end{aligned}$$

1.6 Invariance

1.6.1 Invariant Functions

Definition 1.7 If F is an infinitely differentiable function and $x^* = \Lambda(x;\varepsilon)$ is Lie group of transformations of one parameter ε, then F is invariant function if and only if,

$$F(x^*) \equiv F(x).$$

Theorem 1.3 [5] A function $F(x)$ is invariant under Lie group of transformation $x^* = \Lambda(x;\varepsilon)$ if and only if

$$XF(x) = 0,$$

where X is the infinitesimal generator accompanying of the symmetry transformation.

Theorem 1.4 [5] Let $x^* = \Lambda(x;\varepsilon)$ be the Lie group of transformation, then the identity

$$F(x^*) = F(x) + \varepsilon.$$

holds if and only if $XF(x) \equiv 1$.

1.6.2 Invariant PDE

Let $S^\beta(x;u)$ be a system of q PDEs of order k with n independent variables $x = (x_1, \ldots, x_n)$ and m dependent variables $u(x) = (u_1(x), \ldots, u_m(x))$, given by

$$S^\beta(x, u; \partial u, \ldots, \partial^k u) = 0, \quad \beta = 1, \ldots, q. \tag{1.10}$$

where $(\partial u, \partial^2 u, \ldots, \partial^k u)$ denote the collections of all first, second,..., kth-order partial derivatives, i.e., $u_i^\beta = D_i(u^\beta), u_{ij}^\beta = D_j D_i(u^\beta), \ldots$ respectively, with the total differentiation operator with respect to x^j given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\gamma \frac{\partial}{\partial u^\gamma} + u_{ii_1}^\gamma \frac{\partial}{\partial u_{i_1}^\gamma} + u_{ii_1i_2}^\gamma \frac{\partial}{\partial u_{i_1i_2}^\gamma} + \ldots .$$

in which the summation convention is used whenever appropriate.

Definition 1.8 [5] The one parameter Lie group of point transformation (1.1) leaves invariant the PDE (1.10), i.e., is a point symmetry admitted by PDE (1.10), if and only if its kth extension leaves invariant the surface (1.10).

Theorem 1.5 (Infinitesimal criterion of invariance under a one-parameter Lie group of point transformations [6]). Let X be the infinitesimal generator of a one-parameter Lie group of point transformations $\Lambda^* = \Lambda(x;\varepsilon)$ and $X^{(k)}$ be its kth extension. Then the transformation $\Lambda^* = \Lambda(x;\varepsilon)$ is a point symmetry of the PDE system (1.10) if and only if for each $\beta = 1, \ldots, q$,

$$X^{(k)}[S^{\beta}(x, u; \partial u, \ldots, \partial^k u)] = 0, \tag{1.11}$$

when

$$S^{\beta}(x, u; \partial u, \ldots, \partial^k u) = 0, \quad \beta = 1, \ldots, q. \tag{1.12}$$

1.6.3 Canonical Coordinates

As a result of the theory of PDEs (see[66]), for a system of equations given by

$$\begin{aligned} Xx &= \sum_{i=1}^{n} \xi^i(\beta) \frac{\partial x}{\partial \beta_i} = 1, \\ X\beta'_k &= \sum_{i=1}^{n} \xi^i(\beta) \frac{\partial \beta'_k}{\partial \beta_i} = 0. \end{aligned}$$

where $i = 1, 2, \ldots, n$ and $k = 2, 3, \ldots, n$, there always exist a nontrivial solution $\{x(\beta_i), \beta'_k(\beta_i)\}$.

In the light of this result, we conclude that there exist coordinates, in which the infinitesimal generator can be maximally simplified, called canonical coordinates.

Theorem 1.6 [5] If $\boldsymbol{y} = (y_1(\boldsymbol{x}), y_2(\boldsymbol{x}), \ldots, y_{n-1}(\boldsymbol{x}))$, is any set of canonical coordinates, then the infinitesimal generator of the one-parameter Lie group of transformation $x^* = \Lambda(x; \varepsilon)$ in these coordinates becomes

$$Y = \frac{\partial}{\partial y_n}.$$

Example 1.8 Given a symmetry generator by

$$X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}. \tag{1.13}$$

Then we can find its canonical coordinates as follows:

we solve the following system of PDEs viz.,

$$\begin{aligned} X(r) &= 0, \\ X(s) &= 1. \end{aligned} \tag{1.14}$$

Thus, the canonical coordinates are $r = \frac{y}{x}$, $s = -\frac{1}{x}$ and hence the generator in terms of these coordinates becomes

$$X = \frac{\partial}{\partial s}.$$

1.7 r-Parameter Lie Group of Transformations

Consider an r-parameter Lie group of transformations given by

$$\boldsymbol{x}^* = \boldsymbol{\Lambda}(\boldsymbol{x}; \boldsymbol{\varepsilon})$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and parameters $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$. and the composition law denoted by

$$\boldsymbol{\varphi}(\boldsymbol{\varepsilon}, \boldsymbol{\delta}) = (\boldsymbol{\phi}_1(\boldsymbol{\varepsilon}, \boldsymbol{\delta}), \boldsymbol{\phi}_2(\boldsymbol{\varepsilon}, \boldsymbol{\delta}), \ldots, \boldsymbol{\phi}_r(\boldsymbol{\varepsilon}, \boldsymbol{\delta})),$$

with $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_r)$, where $\boldsymbol{\varphi}(\boldsymbol{\varepsilon}, \boldsymbol{\delta})$ satisfies the axioms of group with $\boldsymbol{\varepsilon} = 0$, and $\boldsymbol{\varphi}(\boldsymbol{\varepsilon}, \boldsymbol{\delta})$ is assumed to be analytic in its domain of definition.

Definition 1.9 (Infinitesimal Generator $\mathbf{X}_i$) The infinitesimal generator $\mathbf{X}_i$, corresponding to the parameter ε_i of r-parameter Lie group of transformation $\mathbf{x}^* = \boldsymbol{\Lambda}(\mathbf{x}; \boldsymbol{\varepsilon})$, is given by

$$\mathbf{X}_i = \sum_{k=1}^{n} \xi_i^k(\mathbf{x}) \frac{\partial}{\partial x_k}, \quad i = 1, 2, \ldots, r. \tag{1.15}$$

1.8 Lie Algebras

Definition 1.10 Let $\mathcal{L}$ be an algebra and its product be defined by $(x, y) \mapsto [x, y]$, then $\mathcal{L}$ is called a Lie algebra if it satisfies the following properties:

- ($\bullet$) $[x, x] = 0$, for all $x \in \mathcal{L}$,

(•) $[x, y] = -[y, x]$, for all $(x, y) \in \mathcal{L}^2$, (**Skew-symmetry**)

(•) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$, $(x, y, z) \in \mathcal{L}^3$, (**Jacobi Identity**).

Definition 1.11 Consider an r-parameter Lie group of transformation $\boldsymbol{x}^* = \boldsymbol{\Lambda}(\boldsymbol{x}; \boldsymbol{\varepsilon})$ with infinitesimal generators $\mathbf{X}_i$ corresponding to each parameter ε_i defined in (1.15). Then, the commutator (*Lie bracket*) of $\mathbf{X}_i$ and $\mathbf{X}_j$ is defined as [56],

$$[\mathbf{X}_i, \mathbf{X}_j] = \mathbf{X}_i\mathbf{X}_j - \mathbf{X}_j\mathbf{X}_i. \tag{1.16}$$

Theorem 1.7 (**Second Fundamental Theorem of Lie [5]**) The commutator of any two infinitesimal generators of an **r**-parameter Lie group of transformation is also an infinitesimal generator. In particular

$$[\boldsymbol{X}_i, \boldsymbol{X}_j] = \sum_{k=1}^{r} C_{ij}^k \boldsymbol{X}_k, \tag{1.17}$$

where the coefficients C_{ij}^k are constants called structure constants, $i, j, k = 1, 2, \ldots, r$.

1.9 Symmetries, Conservation Laws and Double Reductions

In this section, we introduce some preliminaries dealing with symmetries, multiplier approach and double reductions of PDEs.

Definition 1.12 The Lie-Bäcklund (generalized operator) is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\gamma \frac{\partial}{\partial u^\gamma}, \quad \xi^i, \eta^\gamma \in \mathcal{A}, \tag{1.18}$$

where $\mathcal{A}$ is the vector space of differential functions. The operator (1.18) is an abbreviated from infinite formal sum

$$X = \xi^i(x,u) \frac{\partial}{\partial x^i} + \eta^\gamma \frac{\partial}{\partial u^\gamma} + \sum_{s \geq 1} \xi^\gamma_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u^\gamma_{i_1 i_2 \ldots i_s}}, \tag{1.19}$$

where the additional coefficients are determined uniquely by prolongation formula [5]

$$\begin{aligned} \xi^\gamma_i &= D_i(W^\gamma) + \xi^j u^\gamma_{ij}, \\ \xi^\gamma_{i_1 i_2 \ldots i_s} &= D_{i_1 \ldots i_s}(W^\gamma) + \xi^j u^\gamma_{j i_1 \ldots i_s}, \quad s > 1 \end{aligned} \tag{1.20}$$

In (1.20), W^γrepresents the Lie characteristic function

$$W^\gamma = \eta^\gamma - \xi^j u^\gamma_j. \tag{1.21}$$

1.9.1 Multiplier Approach

In this subsection, we introduce the recently established approach, called a multiplier approach, that leads to a large class of conserved quantities which would not be provided by variational techniques or the standard methods especially the higher-order multipliers. This approach is set up based upon the fact that the

Euler-Lagrange operator annihilates a total divergence [23]. In the following, we give a brief description of this approach.

Definition 1.13 Consider $Q_\beta(x, u, \partial u, \dots)$ is a nontrivial differential function and the system of PDE given by Eq.(1.10), we say that Q_β is a multiplier if it satisfies the following:

$$\mathcal{E}_u[Q_\beta S^\beta] = 0. \tag{1.22}$$

where $\mathcal{E}_u$ is the respective Euler-Lagrange operator defined as

$$\mathcal{E}_u = \frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\dots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}, \quad \alpha = 1, \dots, m. \tag{1.23}$$

Definition 1.14 A current vector $T = (T^1, T^2, \dots, T^n),\ T^i \in A, i = 1, \dots, n$ is conserved if it satisfies

$$D_i T^i\big|_{(1.10)} = 0 \tag{1.24}$$

Definition 1.15 Let Q_β be a multiplier satisfying (1.22), then $Q_\beta S^\beta$ is a total divergence, i.e.,

$$Q_\beta S^\beta = D_i T^i, \tag{1.25}$$

for some (conserved) vector T^i. Thus, each multiplier obtained by solving (1.22) leads to the conserved vector determined by the technique given in [31, 19] .

Example 1.9 Illustrative Example (Conserved Vector using Multiplier approach)

Consider the following problem:

$$u_{tt} - u_{xx} - u = 0 \tag{1.26}$$

We want to determine the conserved vector (T^t, T^x) of (1.26). To this end, we find the multiplier Q by applying the Euler-Lagrange Operator on the above equation as follows:

$$\frac{\delta}{\delta u}(Q(u_{tt} - u_{xx} - u)) = 0 \tag{1.27}$$

Since the differential equation is of second order, we take Q is of order upto second derivatives, i.e., $Q = Q(x, t, u, u_x, u_t, u_{xx}, u_{xt})$. The Euler-Lagrange Operator in this case has the following form:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_xD_x\frac{\partial}{\partial u_{xx}} + D_xD_t\frac{\partial}{\partial u_{xt}} + D_tD_t\frac{\partial}{\partial u_{tt}}.$$

where $D_i, i = x, t$, represent the total derivative given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + u_{txx}\frac{\partial}{\partial u_{xx}} + u_{txt}\frac{\partial}{\partial u_{xt}} + u_{ttt}\frac{\partial}{\partial u_{tt}}.$$
$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + u_{xxx}\frac{\partial}{\partial u_{xx}} + u_{xtx}\frac{\partial}{\partial u_{xt}} + u_{xtt}\frac{\partial}{\partial u_{tt}}.$$

Applying the above to Eq.(1.27), leads to tedious calculations (see Appendix A).

Solving the system, we obtain nontrivial multiplier Q given by

$$Q = (c_1 t + c_2)u_x + (c_1 x + c_3)u_t + c_4 e^{(-c(t+x))} - c_5 e^{(\frac{t-x}{4c})} \tag{1.28}$$

Choosing $c_2 \neq 0$, then $Q_1 = u_x$. The next step is to construct the corresponding conserved vector of Q_1 that obtained by solving the following:

$$u_x(u_{tt} - u_{xx} - u) = D_t T^t + D_x T^x$$

where $T^t = T^t(x, t, u, u_x, u_t),\ T^x = T^x(x, t, u, u_x, u_t)$. This again requires tedious calculations of which the result is presented here viz.,

$$\begin{aligned} T^t &= u_x u_t + \alpha(t)u_x + \beta(x,t), \\ T^x &= -\frac{1}{2}u_t^2 - \alpha(t)u_t - \frac{1}{2}u_x^2 - \frac{1}{2}u^2 - \alpha(t)u + a(x,t). \end{aligned} \tag{1.29}$$

such that $a_x + \beta_t = 0$.

1.9.2 Double Reductions

Definition 1.16 [33, 75] A Lie-Bäcklund symmetry generator X of the form (1.19) is associated with a conserved vector T of the system (1.10) if X and T satisfy the relations

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0,\ i = 1, \ldots, n. \tag{1.30}$$

Theorem 1.8 [34, 64, 65] Suppose that X is a Lie-Bäcklund symmetry of (1.10) and $T^i,\ i = 1, \ldots, n$, are the components of the conserved vector of (1.10). Then

$$T^{*i} = [T^i, X] = X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0,\ i = 1, \ldots, n. \tag{1.31}$$

constitute the components of a conserved vector of (1.10).

Theorem 1.9 [10] Suppose that $D_i T^i = 0$ is a conservation law of the PDE system (1.10). Then under a contact transformation, there exist functions $\tilde{T}^i$ such that $J D_i T^i = \tilde{D}_i \tilde{T}^i$, where $\tilde{T}$ is given as

$$\begin{gathered}
\begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \\
J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix}
\end{gathered} \tag{1.32}$$

in which

$$A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \dots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \dots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \dots & \tilde{D}_n x_n \end{pmatrix}, \ A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \dots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \dots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \dots & D_n \tilde{x}_n \end{pmatrix} \tag{1.33}$$

and $J = det(A)$.

Theorem 1.10 [10] **(Fundamental Theorem on Double Reduction)** Suppose that $D_i T^i = 0$ is conservation law of the PDE system (1.10). Then under a similarity transformation of a symmetry X of the form (1.19) for the PDE, there exist functions $\tilde{T}^i$ such that X is still a symmetry for the PDE, satisfying $\tilde{D}_i \tilde{T}^i = 0$ and

$$\begin{pmatrix} X\tilde{T}^1 \\ X\tilde{T}^2 \\ \vdots \\ X\tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} [T^1, X] \\ [T^2, X] \\ \vdots \\ [T^n, X] \end{pmatrix} \tag{1.34}$$

where

$$A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \dots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \dots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \dots & \tilde{D}_n x_n \end{pmatrix}, \ A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \dots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \dots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \dots & D_n \tilde{x}_n \end{pmatrix} \tag{1.35}$$

and $J = det(A)$.

Example 1.10 Illustrative Example[9]

Consider

$$u_{tt} = u(u_{xx} + u_{yy}) + u_x^2 + u_y^2 \tag{1.36}$$

One conserved vector of (1.36) is $T = (-u_t, uu_x, uu_y)$. The Eq.(1.36) admits the following symmetries:

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y}, \\ &X_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \end{aligned} \tag{1.37}$$

The double reduction of (1.36) can be calculated using T as follows:

Using formula (1.30), one can see that X_1, X_2, and X_3 are associated with T, then combine the above symmetries i.e., $X = \frac{\partial}{\partial t} + c_1\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial y}$. The canonical form of X is $\frac{\partial}{\partial q}$ when

$$r = y - c_2 t,\ s = x - c_1 t,\ q = t,\ w(r, s) = u. \tag{1.38}$$

Using the formula (1.32), we obtain the reduced conserved form

$$D_r T^r + D_s T^s = 0, \tag{1.39}$$

where

$$
\begin{aligned}
T^r &= c_2^2 w_r + c_2 c_1 w_s - w w_r, \\
T^s &= c_1 c_2 w_r + c_1^2 w_s - w w_s, \\
Tq &= -c_2 w_r - c_1 w_s.
\end{aligned}
\tag{1.40}
$$

The reduced conserved form admits the inherited symmetry:

$$
\tilde{X}_4 = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}, \tag{1.41}
$$

Similarly by using (1.30), one can show that $\tilde{X}_4$ is associated symmetry. The canonical form of this symmetry is $Y = \frac{\partial}{\partial m}$ and hence the canonical coordinates are

$$
n = \frac{s}{r},\ m = ln(r),\ v(n) = w. \tag{1.42}
$$

Thus, the reduced conserved form using (1.32), is

$$
D_n T^n = 0. \tag{1.43}
$$

where

$$
\begin{aligned}
T^n &= v_n(-c_2^2 n^2 + 2c_2 c_1 n + n^2 v - c_1^2 + v), \\
T^m &= -v_n(-c_2^2 n + c_2 c_1 + nv).
\end{aligned}
\tag{1.44}
$$

Consequently, the second step of double reduction is

$$v_n(-c_2^2n^2 + 2c_2c_1n + n^2v - c_1^2 + v) = C, \tag{1.45}$$

where C is a constant, $n = \frac{x-c_1t}{y-c_2t}$ and $v = u$.

CHAPTER 2

A Symmetry Classification, Reductions and Exact Solutions of Non-linear (2+1) Cylindrical Fin Equation

2.1 Introduction

This chapter is devoted to provide a complete symmetry classification of the non-linear (2+1) fin equation in cylindrical coordinates given by the following governing equation:

$$\frac{1}{x}\frac{\partial}{\partial x}(xk(u)u_x) + \frac{1}{x}\frac{\partial}{\partial y}(\frac{1}{x}k(u)u_y) - N^2 f(x)u = u_t. \tag{2.1}$$

where $k(u)$ and $f(x)$ respectively represent thermal conductivity and heat transfer coefficients. In order to deal with above equation, we put it in an appropriate form:

$$x^2k(u)u_{xx} + x^2k(u)_u u_x^2 + xk(u)u_x + k(u)_u u_y^2 + k(u)u_{yy} - x^2N^2f(x)u = x^2u_t. \tag{2.2}$$

Assuming a radial variable heat transfer coefficient and temperature dependent thermal conductivity, a complete classification of these two functions is obtained via Lie symmetry analysis. Using two dimensional Lie subalgebras of the symmetry generators of the equation, we carry out reduction of the fin equation and whenever possible exact solutions are obtained.

2.2 Symmetry Analysis of the Fin Equation

In order to classify solutions of the Fin equation (2.1), we use the well known Lie symmetry method [8]. This method is based upon finding Lie point symmetries of the PDEs that leave them invariant. Lie group of point transformations of one parameter ϵ under which Eq.(2.1) remains invariant are given as:

$$\begin{aligned}\tilde{x} &= x + \epsilon\xi(x, y, t, u) + O(\epsilon^2),\\ \tilde{y} &= y + \epsilon\eta(x, y, t, u) + O(\epsilon^2),\\ \tilde{t} &= t + \epsilon\tau(x, y, t, u) + O(\epsilon^2),\\ \tilde{u} &= u + \epsilon\phi(x, y, t, u) + O(\epsilon^2),\end{aligned}$$

The corresponding symmetry generator of above transformation is given by [56]:

$$X = \xi(x, y, t, u)\frac{\partial}{\partial x} + \eta(x, y, t, u)\frac{\partial}{\partial y} + \tau(x, y, t, u)\frac{\partial}{\partial t} + \phi(x, y, t, u)\frac{\partial}{\partial u},$$

Since we are dealing with a second order PDE, the above generator needs to be prolonged to include second order derivatives. Then the prolonged generator of

X can be found using the following formula [56]:

$$X^{(2)} = X + \sum_{I=0}^{2} \phi^I \frac{\partial}{\partial u_J} + \sum_{I,J=0}^{2} \phi^{IJ} \frac{\partial}{\partial u_{IJ}}, \qquad I, J = 0, \dots 2, \tag{2.3}$$

with 0 representing t and 1 & 2 representing x & y respectively and where the coefficients, ϕ^J and ϕ^{JK}, of the derivatives with respect to dependent variables in Eq.(2.3) are to be evaluated using the expressions:

$$\phi^J = D_i(\phi - \xi^j u_{ji}) + \xi^j u_{j,i}, \tag{2.4}$$

and

$$\phi^{JK} = D_i D_j(\phi - \xi^j u_{ji}) + \xi^k u_{k,ij}. \tag{2.5}$$

At this stage we use the Lie symmetry criterion by requiring that Eq.(2.1) is invariant under the prolonged symmetry generator given in Eq.(2.3) modulu Eq.(2.1). Mathematically, this requirements is given by,

$$X^{(2)}[x^2k(u)u_{xx}+x^2k(u)_u u_x^2+xk(u)u_x+k(u)_u u_y^2+k(u)u_{yy}-x^2N^2f(x)u-x^2u_t]|_{Eq.(2.1)} = 0. \tag{2.6}$$

and leads to,

$$\begin{aligned}
&\xi(x,y,t,u)[2xk(u)u_{xx} + 2xk(u)_u u_x^2 + k(u)u_x - 2xN^2 f(x)u \\
&-x^2N^2 f(x)_x u - 2xu_t] + \phi(x,y,t,u)[x^2k(u)_{uu}u_x^2 + xk(u)_u u_x + \\
&k(u)_{uu}u_y^2 + k(u)_u u_{yy} - x^2N^2 f(x) + x^2 k(u)_u u_{xx}] + \\
&\phi^x[2x^2k(u)_u u_x + xk(u)] + 2k(u)_u u_y \phi^y - x^2\phi^t + \phi^{xx}x^2k(u) + \phi^{yy}k(u) = 0
\end{aligned} \tag{2.7}$$

Substituting

$$\begin{aligned}
&u_t = k(u)u_{xx} + k(u)_u u_x^2 + \tfrac{1}{x}k(u)u_x + \\
&\tfrac{1}{x^2}k(u)_u u_y^2 + \tfrac{1}{x^2}k(u)u_{yy} - N^2 f(x)u
\end{aligned} \tag{2.8}$$

and the expressions for ϕ^t, ϕ^x, ϕ^y, ϕ^{xx} and ϕ^{yy} using Eqs.(2.4) & (2.5) into Eq. (2.7) and comparing terms involving derivatives of the dependent function u. This gives the following over determined system of linear PDEs :

$$\xi_u = 0 = \eta_u = \tau_u = \phi_{uu} = \tau_y = \tau_x, \tag{2.9}$$

$$x^2\eta_t - k(u)x\eta_x - k(u)x^2\eta_{xx} - k(u)\eta_{yy} + 2k(u)\phi_{uy} = 0, \tag{2.10}$$

$$\begin{aligned} &-k(u)\xi + x\phi(k(u))_u - k(u)x\xi_x + k(u)x\tau_t + x^2\xi_t \\ &+2k(u)x^2\phi_{xu} - k(u)x^2\xi_{xx} - k(u)\xi_{yy} = 0, \end{aligned} \tag{2.11}$$

$$\begin{aligned} &f(x)N^2x^2\phi + N^2ux^2\xi(f(x))_x + x^2\phi_t - f(x)N^2ux^2\phi_u - k(u)x\phi_x + \\ &f(x)N^2ux^2\tau_t - k(u)x^2\phi_{xx} - k(u)\phi_{yy} = 0, \end{aligned} \tag{2.12}$$

$$x^2\eta_x + \xi_y = 0, \tag{2.13}$$

$$\phi k(u)_u - 2k(u)\xi_x + k(u)\tau_t = 0, \tag{2.14}$$

$$-2k(u)\xi + x\phi k(u)_u - 2xk(u)\eta_y + k(u)x\tau_t = 0. \tag{2.15}$$

To determine the unknowns ξ, η, τ and ϕ, we solve the above coupled system of differential equations by first considering Eq.(2.14). Differentiating, this equation twice with respect to u leads to the following expression:

$$\phi_{uu} = (\tfrac{k}{k_u})_{uu}(2\xi_x - \tau_t). \tag{2.16}$$

Using Eq.(2.9) into Eq.(2.16), we get,

$$(\tfrac{k}{k_u})_{uu}(2\xi_x - \tau_t) = 0. \tag{2.17}$$

We proceed from above equation to obtain complete classification of both $k,$ and f as shown in the next section.

2.3 Classification

In order to find a complete classification of solutions of Eq.(2.1), we note that the following three cases arise from Eq.(2.17):

(I) $(\frac{k}{k_u})_{uu} = 0,$

(II) $2\xi_x - \tau_t = 0,$

(III) $2\xi_x - \tau_t = 0 = (\frac{k}{k_u})_{uu}.$

For obtaining a complete classification, we consider all the three cases one by one. Since procedure of classification in all the three cases is similar, we give detailed procedure in the first case, and only give results in the remaining cases. To begin the classification, we proceed as follows:

3.1 Case I

Solving equation $(\frac{k}{k_u})_{uu} = 0$, instantly yields,

$$k(u) = \gamma(\alpha u + \beta)^{\frac{1}{\alpha}}, \tag{2.18}$$

where γ, α and β are integration constants. Using Eq.(2.18) into Eq.(2.14), immediately gives,

$$\phi = (\alpha u + \beta)(2\xi_x - \tau_t). \tag{2.19}$$

Using Eqs.(2.18) & (2.19) in Eq.(2.15), we obtain a differential relation in ξ and η given by,

$$\eta_y = \xi_x - \tfrac{1}{x}\xi. \tag{2.20}$$

Differentiating Eq.(2.19) twice, and Eq.(2.20) once w.r.t y and using Eq.(2.13) in the resulting expressions gives,

$$\phi_{uy} = -2\alpha x^2\eta_{xx} - 4\alpha x\eta_x, \tag{2.21}$$

and

$$\eta_{yy} = -x^2\eta_{xx} - x\eta_x. \tag{2.22}$$

At this stage we use Eqs.(2.22) and (2.21) into Eq.(2.10), to get,

$$x^2\eta_t - 4\alpha k x^2\eta_{xx} - 8\alpha k x\eta_x = 0. \tag{2.23}$$

Differentiating Eq.(2.23) w.r.t u, while keeping Eq.(2.18) in mind, we obtain,

$$-4\alpha\gamma(\alpha u + \beta)^{(\frac{1}{\alpha}-1)}x[x\eta_{xx} + 2\eta_x] = 0. \tag{2.24}$$

Keeping in mind that α and γ in the above equation are non-zero, we conclude that the above equation is satisfied only when

$$x\eta_{xx} + 2\eta_x = 0. \tag{2.25}$$

(the case $\alpha = 1$ is not be considered as it becomes a special case of (I.c) that is dealt with later). Note that Eq.(2.25) is a separable DE and can be easily solved to find η given by,

$$\eta(x, y, t) = -c(y, t)x^{-1} + d(y, t). \tag{2.26}$$

To require consistency of η, we use Eq.(2.26) into Eq.(2.23). It suggests that Eq.(2.23) is satisfied when the following differential constraint is met,

$$-c_t(y, t) + xd_t(y, t) = 0. \tag{2.27}$$

The above equation implies that $d_t(y, t) = \gamma(y)$ and $c(y, t) = \lambda(y)$. Using these results into Eq.(2.27) gives,

$$\eta(x, y, t) = -\lambda(y)x^{-1} + \gamma(y). \tag{2.28}$$

To determine γ we use Eq.(2.28) into Eq.(2.22). This leads to the following second order differential constraint,

$$x\gamma_{yy}(y) - \lambda_{yy}(y) - \lambda(y) = 0. \tag{2.29}$$

To solve the above equation, we differentiate it with respect to x to first get,

$$\gamma(y) = c_1 y + c_2. \tag{2.30}$$

We put the value of η given by Eq.(2.30) into Eq.(2.29), to get a second order linear differential equation,

$$\lambda_{yy}(y) + \lambda(y) = 0, \tag{2.31}$$

The above equation can be easily solved to get,

$$\lambda(y) = c_3 \cos y + c_4 \sin y. \tag{2.32}$$

At this stage we use Eqs.(2.30) and (2.32) into Eq.(2.28), to infer that,

$$\eta(x, y) = \tfrac{-1}{x}[c_3 \cos y + c_4 \sin y] + c_1 y + c_2. \tag{2.33}$$

Having determined η completely, we now find ξ. For this purpose, we use Eq.(2.33) there. This yields an expression for ξ as given below:

$$\xi(x, y, t) = -c_3 \sin y + c_4 \cos y + \gamma_1(x, t). \tag{2.34}$$

To determine γ_1, we use Eqs.(2.34) and (2.33) into Eq.(2.20). This gives,

$$\gamma_1(x, t) = c_1 x \ln x + x\lambda_1(t). \tag{2.35}$$

Using above value of γ_1 into Eqs.(2.34) and (2.19), respectively become:

$$\xi(x,y,t) = -c_3 \sin y + c_4 \cos y + c_1 x \ln x + x\lambda_1(t), \tag{2.36}$$

and

$$\phi = (\alpha u + \beta)(2c_1 + 2c_1 \ln x + 2\lambda_1(t) - \tau_t). \tag{2.37}$$

We now use all the above results into Eq.(2.11), which is satisfied if the following condition is met:

$$x^3\lambda_1(t) + 4\gamma\alpha c_1(\alpha u + \beta)^{\frac{1}{\alpha}} = 0. \tag{2.38}$$

From the above equation, we immediately infer that $\lambda_1(t) = c_5$ and $c_1 = 0$. Therefore, Eqs.(2.33), (2.36) and (2.37) take the form,

$$\begin{aligned}
&\xi(x,y,t) = -c_3 \sin y + c_4 \cos y + c_5 x,\\
&\eta(x,y) = \tfrac{-1}{x}[c_3 \cos y + c_4 \sin y] + c_2,\\
&\phi = (\alpha u + \beta)(2c_5 - \tau_t).
\end{aligned} \tag{2.39}$$

To check consistency, we now use the remaining equation, i.e, Eq.(2.12). This requirement gives:

$$\begin{aligned}
&\beta f(x) N^2 (2c_5 - \tau_t) + N^2 u f_x(-c_3 \sin y + c_4 \cos y + c_5 x)\\
&-(\alpha u + \beta)\tau_{tt} + f N^2 u \tau_t = 0.
\end{aligned} \tag{2.40}$$

In order that Eq.(2.40) is satisfied, we proceed as follows: Comparing coefficients of $f(x)$, this equation gives,

$$-\beta f(x)N^2\tau_{tt} - (\alpha u + \beta)\tau_{ttt} + f(x)N^2 u\tau_{tt} = 0. \tag{2.41}$$

Differentiating above equation with respect to u gives,

$$-\alpha\tau_{ttt} + f(x)N^2\tau_{tt} = 0, \tag{2.42}$$

which is a separable DE in x and t, and can be written as

$$\frac{\tau_{ttt}}{\tau_{tt}} = \frac{f(x)N^2}{\alpha}. \tag{2.43}$$

Solving Eq.(2.43) implies that $f(x) = c$ (c a constant) whereas the τ becomes,

$$\tau(t) = \frac{c_6\alpha^2}{c^2N^4}\exp(\frac{cN^2}{\alpha}t) + c_7t + c_8. \tag{2.44}$$

To require consistency, we use Eq.(2.44) with $f(x) = c \neq 0$ in Eq.(2.41), to obtain

$$\beta c_6(\alpha + 1) = 0. \tag{2.45}$$

From Eq.(2.45) four subcases arise, namely,

(I.a.) $\alpha = -1, \beta \neq 0$, $c_6 \neq 0$,

(I.b.) $\alpha > 0, \beta = 0$, $c_6 \neq 0$, (I.c.) $\alpha > 0, \beta \neq 0$, $c_6 = 0$,

(I.d.) $\alpha = -1, \beta = 0$, $c_6 = 0$.

We first consider I.a.

3.1.1. Subcase (I.a.) $k(u) = \dfrac{\gamma}{\beta - u}$ and $f(x) = c$

Using theses conditions arising in this case into Eqs.((2.44),(2.40) and (2.39)) the infinitesimal symmetry generators ξ, η, τ, ϕ and k are determined:

$$\begin{aligned}
&\xi = -c_3 \sin y + c_4 \cos y, \\
&\eta = -c_3 \tfrac{\cos y}{x} - c_4 \tfrac{\sin y}{x} + c_2, \\
&\tau = \tfrac{c_6}{c^2 N^4} \exp(-cN^2 t) + c_8, \\
&\phi = \tfrac{c_6}{cN^2} (-u + \beta) \exp(-cN^2 t).
\end{aligned} \tag{2.46}$$

The $five$ symmetry generators associated with above infinitesimals are given by,

$$\begin{aligned}
&X_1 = -\sin y \frac{\partial}{\partial x} - \frac{\cos y}{x} \frac{\partial}{\partial y},\ X_2 = \cos y \frac{\partial}{\partial x} - \frac{\sin y}{x} \frac{\partial}{\partial y} \\
&X_3 = \frac{1}{c^2 N^4} \exp(-cN^2 t) \frac{\partial}{\partial t} + \frac{\beta - u}{cN^2} \exp(-cN^2 t) \frac{\partial}{\partial u}, \quad X_4 = \frac{\partial}{\partial t},\ X_5 = \frac{\partial}{\partial y}.
\end{aligned} \tag{2.47}$$

The commutation relation for each of the above symmetry generators are listed in table (2.1).

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	X_2
X_2	0	0	0	0	X_1
X_3	0	0	0	cN^2X_3	0
X_4	0	0	$-cN^2X_3$	0	0
X_5	$-X_2$	X_1	0	0	0

Table 2.1: Case I.a: Commutation relations satisfied by symmetry generators

3.1.2. Subcase (I.b.) $k(u) = \gamma(\alpha u)^{\frac{1}{\alpha}}$ and $f(x) = c$

Using the values of k and f arising in this case into Eqs.((2.44),(2.40) and (2.39)), the expressions for $\xi, \eta, \tau,$ and ϕ takes the form:

$$\begin{aligned} &\xi = -c_3 \sin y + c_4 \cos y + c_5 x, \quad \eta = -c_3 \frac{\cos y}{x} - c_4 \frac{\sin y}{x} + c_2, \\ &\tau = \frac{c_6 \alpha^2}{c^2 N^4} \exp(-cN^2 t) + c_8, \quad \phi = \alpha u (2c_5 - \frac{c_6 \alpha}{cN^2} \exp(-cN^2 t)). \end{aligned} \tag{2.48}$$

Accordingly, the *six* symmetry generators associated with above infinitesimals are given by,

$$\begin{aligned} &X_1 = -\sin y \frac{\partial}{\partial x} - \frac{\cos y}{x} \frac{\partial}{\partial y}, \quad X_2 = \cos y \frac{\partial}{\partial x} - \frac{\sin y}{x} \frac{\partial}{\partial y}, \\ &X_3 = x \frac{\partial}{\partial x} + 2\alpha u \frac{\partial}{\partial u}, \quad X_4 = \frac{\alpha^2}{c^2 N^4} \exp(\frac{cN^2}{\alpha} t) \frac{\partial}{\partial t} - \frac{\alpha^2 u}{cN^2} \exp(\frac{cN^2}{\alpha} t) \frac{\partial}{\partial u}, \\ &X_5 = \frac{\partial}{\partial t}, \quad X_6 = \frac{\partial}{\partial y}. \end{aligned} \tag{2.49}$$

The commutation relation for generators given by Eq.(2.49).

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	X_1	0	0	X_2
X_2	0	0	X_2	0	0	$-X_1$
X_3	$-X_1$	$-X_2$	0	0	0	0
X_4	0	0	0	0	$\frac{cN^2}{\alpha}X_4$	0
X_5	0	0	0	$\frac{-cN^2}{\alpha}X_4$	0	0
X_6	$-X_2$	X_1	0	0	0	0

Table 2.2: Commutation relations in case I.b

3.1.3. Subcase (I.c.) $k(u) = \gamma(\alpha u + \beta)^{\frac{1}{\alpha}}$ and $f(x) = c$

Using the conditions with the values of$k$ and f of this case into Eqs.((2.44),(2.40) and (2.39)), the expressions for $\xi, \eta, \tau,$ and ϕ takes the form:

$$\begin{aligned}
&\xi = -c_3 \sin y + c_4 \cos y, \quad \eta = -c_3 \frac{\cos y}{x} - c_4 \frac{\sin y}{x} + c_2, \\
&\tau = c_8, \quad \phi = 0.
\end{aligned} \tag{2.50}$$

The corresponding generators are,

$$\begin{aligned}
&X_1 = -\sin y \frac{\partial}{\partial x} - \frac{\cos y}{x}\frac{\partial}{\partial y}, \quad X_2 = \cos y \frac{\partial}{\partial x} - \frac{\sin y}{x}\frac{\partial}{\partial y}, \\
&X_3 = \frac{\partial}{\partial t}, \quad X_4 = \frac{\partial}{\partial y}.
\end{aligned} \tag{2.51}$$

As before the commutation relations form a closed algebra and are given in the following:

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	X_2
X_2	0	0	0	$-X_1$
X_3	0	0	0	0
X_4	$-X_2$	X_1	0	0

Table 2.3: Commutation relations in case I.c

3.1.4. Subcase (I.d.) $k(u) = -\dfrac{\gamma}{u}$ and $f(x) = c$

In this case Eqs.(2.44),(2.40) and (2.39), result in the following expressions for $\xi, \eta, \tau,$ and ϕ:

$$
\begin{aligned}
&\xi = -c_3 \sin y + c_4 \cos y + c_5 x, \quad \eta = -c_3 \frac{\cos y}{x} - c_4 \frac{\sin y}{x} + c_2, \\
&\tau = c_8, \quad \phi = -2c_5 u.
\end{aligned}
\tag{2.52}
$$

With above infinitesimals there are five generators associated:

$$
\begin{aligned}
&X_1 = -\sin y \frac{\partial}{\partial x} - \frac{\cos y}{x} \frac{\partial}{\partial y}, \quad X_2 = \cos y \frac{\partial}{\partial x} - \frac{\sin y}{x} \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u} \\
&X_4 = \frac{\partial}{\partial t}, \quad X_5 = \frac{\partial}{\partial y}.
\end{aligned}
\tag{2.53}
$$

The commutation relation satisfied by the above *five* generators is given in table (2.4).

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	X_1	0	X_2
X_2	0	0	X_2	0	$-X_1$
X_3	$-X_1$	$-X_2$	0	0	0
X_4	0	0	0	0	0
X_5	$-X_2$	X_1	0	0	0

Table 2.4: Commutation relations in case I.d

3.2 Case II ($2\xi_x - \tau_t = 0.$)

In this case the system of determining equations given by Eqs. ((2.9)-(2.15)) becomes,

$$\begin{aligned}
&\xi_u = 0 = \eta_u = \tau_u = \phi = \tau_y = \tau_x, \\
&x^2\eta_t - k(u)x\eta_x - k(u)x^2\eta_{xx} - k(u)\eta_{yy} = 0, \\
&-k(u)\xi + k(u)x\xi_x + x^2\xi_t - k(u)x^2\xi_{xx} - k(u)\xi_{yy} = 0, \\
&N^2ux^2\xi(f(x))_x + 2f(x)N^2ux^2\xi_x = 0, \\
&x^2\eta_x + \xi_y = 0, \\
&-2k(u)\xi - 2xk(u)\eta_y + 2k(u)x\xi_x = 0.
\end{aligned} \tag{2.54}$$

Following the procedure adopted in case I, we easily find that,

$$\xi = c_1 x, \quad \eta = c_3, \quad \tau = 2c_1 + c_2, \quad \phi = 0. \tag{2.55}$$

The above infinitesimals satisfy all the equations in the system (2.54) except Eq.(iv). Using Eq.(2.55) in ((2.54)-(iv))[1], we obtain,

$$c_1 N^2 x^3 u f_x + 2c_1 N^2 x^2 u f = 0. \tag{2.56}$$

From Eq.(2.56) two cases arise:

(II.a) $c_1 = 0$,

(II.b) $c_1 \neq 0$.

3.2.1 Subcase II.a

In this case $k(u)$ and $f(x)$ in system (2.54) are arbitrary functions, and the general expression of ξ, η, τ and ϕ takes the form,

$$\xi = 0, \quad \eta = c_4, \quad \tau = c_5, \quad \phi = 0. \tag{2.57}$$

The symmetry generators in this case are,

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}. \tag{2.58}$$

3.2.2 Case II.b

Here $k(u)$ is an arbitrary functions and $f(x) = \dfrac{c}{x^2}$. The general expression of ξ, η, τ and ϕ are,

$$\tau = 2c_1 t + c_2, \quad \xi = c_1 x, \quad \eta = c_3, \quad \phi = 0. \tag{2.59}$$

The *three* symmetry generators associated with Eq.(2.59) are,

$$X_1 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial y}. \tag{2.60}$$

The commutation relations satisfied by three generators are presented in the table given below.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	$-2X_2$	0
X_2	$2X_2$	0	0
X_3	0	0	0

Table 2.5: Commutation relations in case II.b

3.3 Case III ($2\xi_x - \tau_t = 0 = (\frac{k}{k_u})_{uu}$.)

This case gives,

$$k(u) = \gamma(\alpha u + \beta)^{\frac{1}{\alpha}}, \quad \phi = 0, \quad \tau_t = 2\xi_x. \tag{2.61}$$

Consequently, the system of determining equations given by Eqs.((2.9)-(2.15)) becomes,

$$
\begin{aligned}
&\xi_u = 0 = \eta_u = \tau_u = \phi = \tau_y = \tau_x,\\
&x^2\eta_t - \gamma(\alpha u+\beta)^{\frac{1}{\alpha}}x\eta_x - \gamma(\alpha u+\beta)^{\frac{1}{\alpha}}x^2\eta_{xx} - \gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\eta_{yy} = 0,\\
&-\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi + \gamma(\alpha u+\beta)^{\frac{1}{\alpha}}x\xi_x + x^2\xi_t - \gamma(\alpha u+\beta)^{\frac{1}{\alpha}}x^2\xi_{xx} - \gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi_{yy} = 0,\\
&N^2ux^2\xi(f(x))_x + 2f(x)N^2ux^2\xi_x = 0,\\
&x^2\eta_x + \xi_y = 0,\\
&-2\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi - 2x\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\eta_y + 2\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}x\xi_x = 0.
\end{aligned}
\tag{2.62}
$$

Remark 2.1 Following the procedure adopted in earlier cases for the system (2.62), we obtain same symmetry generators in cases (3.2 case II) and (3.3 case III). The difference though is that in (3.2 case II) $k(u)$ is arbitrary while in (3.3 case III) $k(u) = \gamma(\alpha u+\beta)^{\frac{1}{\alpha}}$.

2.4 Reduction under two dimensional subalgebra and exact invariant solutions

2.4.1 Case 1

In this section, we present solutions of Eq.(2.1) via reductions. These reductions are obtained by the similarity variables obtained through symmetry generators. To perform reductions of Eq.(2.1), we first consider two symmetry generators, from table (2.1) X_1, and X_2 which span an abelian subalgebra. To start reduction, we first consider X_1. The characteristic equation corresponding to this generator,

$$\frac{dx}{-\sin y} = \frac{-xdy}{\cos y} = \frac{dt}{0} = \frac{du}{0}. \tag{2.63}$$

Solving the above equation it is straight forward [5] to find that it yields the similarity variables, $r = x\cos y$ and $s = t$ with $w(r,s) = u$. Replacing u in terms of new variables Eq.(2.1) becomes,

$$\frac{\gamma}{\beta - w} w_{rr} + \frac{\gamma}{(\beta - w)^2} w_r^2 - N^2 cw - w_s = 0. \tag{2.64}$$

To proceed further, we first transform X_2 in terms of the new variables r , s and w. Thus, $\widehat{X}_2 = \frac{\partial}{\partial r}$. The similarity variables corresponding to this generator are $z = s$ and $v(z) = w$. This reduces Eq.(2.64) to a first-order differential equation given by,

$$v_z + N^2 cv = 0, \tag{2.65}$$

and solving this equation, we immediately find that $v(z) = C_1 \exp(-N^2 cz)$, which is in original coordinates becomes,

$$u(x, y, t) = C_1 \exp(-N^2 ct). \tag{2.66}$$

2.4.2 Case 2

Here, we first consider the generators X_1, and X_3 given in table (2.3), satisfying $[X_1, X_3] = 0$. Following procedure followed in the previous case, the generator X_1 reduces Eq.(2.1) to Eq.(2.64). In the light of X_1, the X_3 transforms to $\widehat{X}_3 = \frac{1}{c^2 N^4} \exp(-cN^2 s)\frac{\partial}{\partial s} + \frac{\beta - w}{cN^2} \exp(-cN^2 s)\frac{\partial}{\partial w}$, which gives $z = r$ with $w = \beta - \exp(-cN^2 s)v(z)$. In the light of these similarity variables, Eq.(2.64) reduces to the following ODE:

$$v_{zz} - \frac{1}{v} v_z^2 + \frac{N^2 c\beta}{\gamma} v = 0. \tag{2.67}$$

Choosing $\gamma = N^2 c\beta$, the solution of the above equation takes the form,

$$v(z) = c_2 \exp(c_1 z - \frac{1}{2} z^2). \tag{2.68}$$

Writing above in original coordinates, it becomes,

$$u(x, y, t) = \beta - c_2 \exp(-N^2 ct) \exp(c_1 x \cos y - \frac{1}{2} x^2 \cos^2 y). \tag{2.69}$$

The graphical profile of the above solution is given in figure (2.1). For constant t the same solution is plotted as shown in figure (2.2).

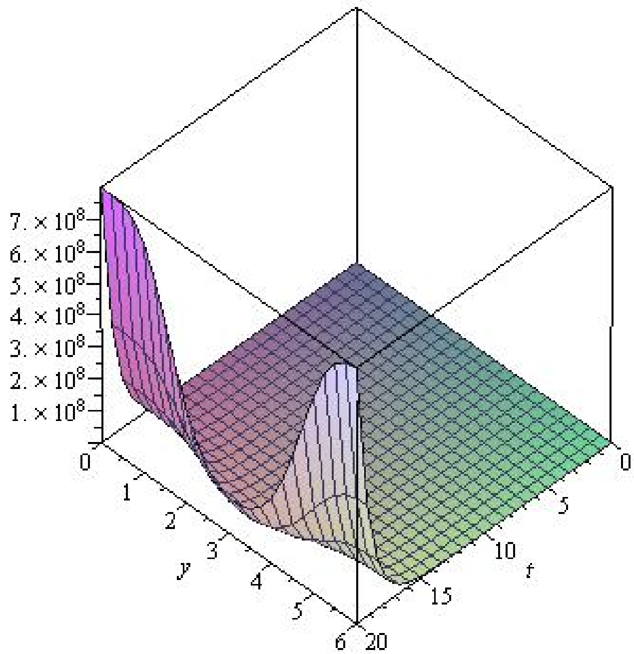

Figure 2.1: Plot of solution (2.69) with $c_1 = c_2 = 1$, $\beta = 0$, $N^2c = -1$, and $x = constant$.

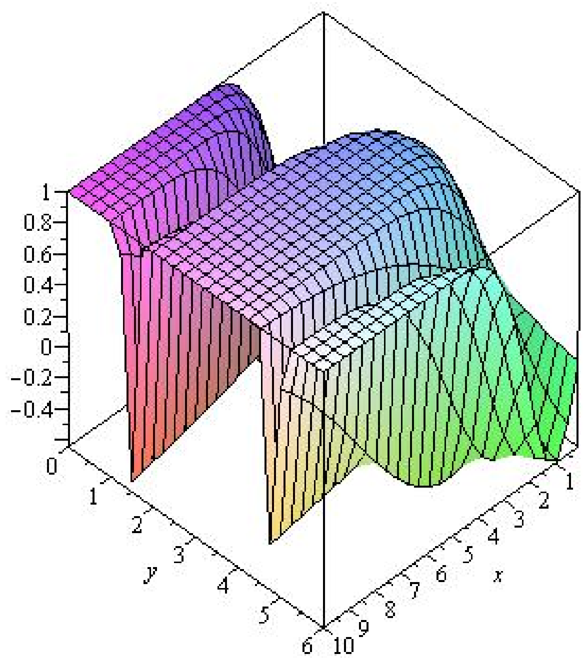

Figure 2.2: Plot of solution in Eq. (2.69) with $c_1 = c_2 = \beta = 1$, $N^2c = -1$, and t=constant.

2.4.3 Case 3

In this case, we consider the two generators X_3, and X_6 that satisfy $[X_3, X_6] = 0$ as shown in table (2.2). Since the two generators commute, we can start reduction by either of X_3 and X_6. First considering X_3, the characteristic equation becomes,

$$\frac{dx}{x} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{2\alpha u}. \tag{2.70}$$

The similarity variables corresponding to above equation become $r = y$, $s = t$ and $u = x^{2\alpha} w$.These variables reduce Eq.(2.1) to a PDE of the form,

$$\begin{aligned} &4\gamma\alpha^{\frac{1}{\alpha}+1}(\alpha+1)w^{\frac{1}{\alpha}+1} + \gamma\alpha^{\frac{1}{\alpha}-1}w^{\frac{1}{\alpha}-1}w_r^2 \\ &+ \gamma\alpha^{\frac{1}{\alpha}}w^{\frac{1}{\alpha}}w_{rr} - N^2cw - w_s = 0. \end{aligned} \tag{2.71}$$

Using similarity variables obtained from X_3, transforms X_6 to $\widehat{X}_6 = \frac{\partial}{\partial r}$. This leads to the new coordinates $s = z$, $v(z) = w$, In the light of these, Eq.(2.71) transforms to,

$$4\gamma\alpha^{\frac{1}{\alpha}+1}(\alpha+1)v^{\frac{1}{\alpha}+1} - N^2cv - v_z = 0. \tag{2.72}$$

Choosing $N^2c = 1$, $\gamma = 1$ and $\alpha = -1$, the above equation takes the form,

$$v_z + 2v = 0. \tag{2.73}$$

giving exact solution $u(x, y, t) = C_1 \exp(2t)$.

2.4.4 Case 4

Here, we consider the generators X_3, X_4 from table (2.4) that satisfy $[X_3, X_4] = 0$. First considering X_3 and its characteristic equation,

$$\frac{dx}{x} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{-2u}. \tag{2.74}$$

gives the similarity variables $r = y$, $s = t$ and $u = x^{-2}w$. These variables reduce Eq.(2.1) to a PDE given below:

$$\gamma w^{-2} w_r^2 - \gamma w^{-1} w_{rr} - N^2 c w - w_s = 0. \tag{2.75}$$

To reduce the above equation further, we use X_3, to transform X_4 to $\widehat{X}_4 = \frac{\partial}{\partial s}$. This leads to the similarity variables $r = z$, $v(z) = w$. Using these , Eq.(2.75) becomes an ODE,

$$v_{zz} - v^{-1} v_z^2 + \frac{N^2 c}{\gamma} v^2 = 0. \tag{2.76}$$

Choosing $\gamma = N^2 c$, Eq.(2.76) can be solved to obtain,

$$v(z) = \frac{1 - \tanh\left(\frac{z+c_2}{2c_1}\right)^2}{c_1^2}. \tag{2.77}$$

Re-casting above in its original coordinates, the exact solution of Eq.(2.1) becomes,

$$u(x,y,t) = \frac{1 - \tanh\left(\frac{y+c_2}{2c_1}\right)^2}{c_1^2 x^2}. \tag{2.78}$$

The graph of this solution is plotted as shown in figure (2.3):

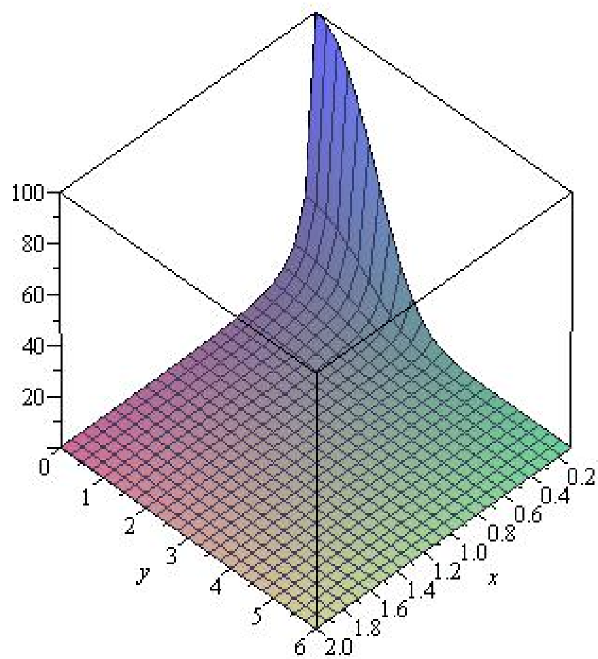

Figure 2.3: Plot of solution given by Eq. (2.78) with $c_1 = 1$, and $c_2 = 0$.

2.4.5 Case 5

Here we consider the symmetry generators X_1, X_3 which satisfy a commutative relationship $[X_1, X_3] = 0$ as shown in table (2.5). First considering X_1, we obtain the similarity variables $r = y$, $s = xt^{-\frac{1}{2}}$ and $w = u$. In these variables, Eq. (2.1) takes the form,

$$\begin{aligned} &s^2k(w)w_{ss} + s^2k_w(w_s)^2 + skw_s + k_w(w_r)^2 \\ &+ kw_{rr} - cN^2w + \frac{1}{2}s^3w_s = 0. \end{aligned} \tag{2.79}$$

First transforming X_3 to $\widehat{X}_3 = \frac{\partial}{\partial r}$, and then solving the resulting characteristic equation we find that $s = z$, $v(z) = w$. These variables can be used to re-cast Eq.(2.79) to an ODE,

$$z^2k(v)v_{zz} + z^2k_v(v_z)^2 + zk(v)v_z - cN^2v + \frac{1}{2}z^3v_z = 0. \tag{2.80}$$

Choosing $k(v) = 1$, one can solve the above equation to get,

$$\begin{aligned} v(z) = {} & C_1 \exp(-\frac{1}{8}z^2)z(BesselI(0, \frac{1}{8}z^2) + BesselI(1, \frac{1}{8}z^2)) \\ & + C_2 \exp(-\frac{1}{8}z^2)z(-BesselK(0, \frac{1}{8}z^2) + BesselK(1, \frac{1}{8}z^2)). \end{aligned}$$

Finally, solution of Eq.(2.1) becomes,

$$\begin{aligned} u(x, y, t) = {} & C_1 \exp(-\frac{1}{8}\frac{x^2}{t})\frac{x}{\sqrt{t}}(BesselI(0, \frac{1}{8}\frac{x^2}{t}) + BesselI(1, \frac{1}{8}\frac{x^2}{t})) \\ & + C_2 \exp(-\frac{1}{8}\frac{x^2}{t})\frac{x}{\sqrt{t}}(-BesselK(0, \frac{1}{8}\frac{x^2}{t}) + BesselK(1, \frac{1}{8}\frac{x^2}{t})). \end{aligned}$$

Remark: Other reductions of (2.1) to ODEs under two-dimensional subalgebras of Lie symmetry generators are given in the Appendix B. .

2.5 Conclusion

A complete classification of the Lie point symmetries of the non-linear fin equation in cylindrical coordinates according to thermal diffusivity and heat transfer coefficient is obtained. Using an exhaustive procedure, the determining equations obtained in the process are completely solved for all possible forms of thermal diffusivity and heat transfer. In all cases reduction of the fin equation is performed. In some cases, the non-linear fin equation is solved for its exact solutions and solutions plotted. As for the symmetry groups are concerned, it is found that the fin equation admits the maximal Lie symmetry group $G < 6 >$ while the minimal Lie symmetry group is $G < 3 >$. The other intermediate groups are $G < 5 >$ and $G < 4 >$. It is hoped that the non-linear fin equation may yield interesting results if the study is extended beyond cylindrical symmetry.

CHAPTER 3

Invariance, Conservation Laws and Exact Solutions of Nonlinear (2+1) Cylindrical Fin Equation

3.1 Introduction

This chapter is concerned with the conservation laws of nonlinear (2+1) fin equation in cylindrical form, viz.,

$$\frac{1}{x}\frac{\partial}{\partial x}(xk(u)u_x) + \frac{1}{x}\frac{\partial}{\partial y}(\frac{1}{x}k(u)u_y) - N^2 f(x)u = u_t. \tag{3.1}$$

The above equation can be rewritten as

$$x^2k(u)u_{xx}+x^2k(u)_u u_x^2+xk(u)u_x+k(u)_u u_y^2+k(u)u_{yy}-N^2x^2f(x)u-x^2u_t = 0. \tag{3.2}$$

The determination of conservation laws is an important topic in studying differential equations, and many authors have worked in this direction [9, 8, 29, 23, 2, 4, 33, 65, 64, 53, 37]. Due to the importance of this topic, many methods have been devised to obtain conserved quantities associated with the differential equations and the so-called conservation laws. We then use the conservation laws to find the exact solutions of Eq.(3.2) using double reduction. In this chapter, we use the method explained in [Ch.1- sec 1.9], called the Multiplier approach, to derive the conservation laws associated with Eq.(3.1).

3.2 Conservation Laws and Double Reduction

In this section, we derive the conservation laws of nonlinear (2+1)-dimensional cylindrical fin equation (3.1) by using the method of multipliers. Then we perform double reduction of PDE and hence obtain exact solution. The conserved vector (T^t, T^x, T^y) of Eq.(3.2) satisfies the divergence relation

$$\begin{aligned} D_tT^t + D_xT^x + D_yT^y = Q\big(&x^2k(u)u_{xx} + x^2k(u)_u u_x^2 \\ &+xk(u)u_x + k(u)_u u_y^2 + k(u)u_{yy} - N^2x^2f(x)u - x^2u_t\big). \end{aligned} \tag{3.3}$$

Thus,

$$\mathcal{E}_u\left[Q\left(x^2k(u)u_{xx} + x^2k(u)_u u_x^2 + xk(u)u_x + k(u)_u u_y^2 + k(u)u_{yy} - N^2x^2f(x)u - x^2u_t\right)\right] = 0, \tag{3.4}$$

where $\mathcal{E}_u$ is the respective Euler-Lagrange operator and Q is called a multiplier. If we suppose Q to be upto second order in derivatives, i.e., $Q = Q(t, x, y, u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, u_{tx}, u_{ty}, u_{xy})$, then application to Eq.(3.4) leads to cumbersome calculations for which the results cannot be presented here. Nevertheless, solving the system, we obtain some nontrivial multipliers Q, each one leads to conserved vector determined by the technique used in [23, 32]. We consider the special case $f(x) = c$ where c is arbitrary constant. In accordance with this choice, we obtain the following forms of Q,

$$Q_1 = -\frac{e^{N^2ct}}{x}, \quad Q_2 = -\frac{e^{N^2ct}}{x}\ln(x). \tag{3.5}$$

The corresponding conserved vector of Q_1 is given by

$$T^1 = (T^t, T^x, T^y) = (e^{N^2t}xu, -e^{N^2t}xk(u)u_x, \frac{-e^{N^2t}k(u)u_y}{x}). \tag{3.6}$$

and Q_2 is given by

$$T^2 = \left(e^{N^2t}x\ln(x)u, \int_0^1 e^{N^2t}(-x\lambda\ln(x)uk'(u)u_x + k(\lambda u)(u - x\ln(x)u_x))\,d\lambda,\right.$$
$$\left.\frac{-e^{N^2t}xk(u)\ln(x)u_y}{x}\right). \tag{3.7}$$

Now we perform the double reduction for particular case $k(u) = \gamma(\alpha u)^{\frac{1}{\alpha}}$, $N^2 = 1$ and $f(x) = 1$. (see[Ch.2- Sec.2.3])

The Eq.(3.2) with the above choice of k, f and N^2 admits the symmetry generators

$$
\begin{aligned}
&X_1 = -\sin(y)\frac{\partial}{\partial x} - \frac{\cos(y)}{x}\frac{\partial}{\partial y}, \quad X_2 = \cos(y)\frac{\partial}{\partial x} - \frac{\sin(y)}{x}\frac{\partial}{\partial y}, \\
&X_3 = x\frac{\partial}{\partial x} + 2\alpha u\frac{\partial}{\partial u}, \quad X_4 = e^{\frac{t}{\alpha}}(\alpha^2\frac{\partial}{\partial t} - \alpha^2 u\frac{\partial}{\partial u}), \\
&X_5 = \frac{\partial}{\partial t}, \quad X_6 = \frac{\partial}{\partial y}.
\end{aligned} \tag{3.8}
$$

Firstly, we show that $\mathbf{X_4}$ is associated with T^1 by using the result

$$
X_4^{[1]}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} - \begin{pmatrix} D_t\tau & D_x\tau & D_y\tau \\ D_t\xi & D_x\xi & D_y\xi \\ D_t\eta & D_x\eta & D_y\eta \end{pmatrix}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} + (D_t\tau + D_x\xi + D_y\eta)\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} = 0. \tag{3.9}
$$

We have

$$
\begin{aligned}
&X_4^{[1]} = X_4 - \alpha^2 e^{\frac{t}{\alpha}} u_x\frac{\partial}{\partial u_x} - \alpha^2 e^{\frac{t}{\alpha}} u_y\frac{\partial}{\partial u_y} \\
&- (\alpha u e^{\frac{t}{\alpha}} + (\alpha^2 e^{\frac{t}{\alpha} - \alpha e^{\frac{t}{\alpha}} u_t})\frac{\partial}{\partial u_t}.
\end{aligned} \tag{3.10}
$$

Calculating the above quantities yield

$$
X_4^{[1]}T^t = 0, \quad X_4^{[1]}T^x = \alpha e^{t-\frac{t}{\alpha}} x(\alpha u)^{\frac{1}{\alpha}} u_x, \quad X_4^{[1]}T^y = \frac{\alpha e^{t-\frac{t}{\alpha}} x(\alpha u)^{\frac{1}{\alpha}} u_y}{x}. \tag{3.11}
$$

and we have $\xi = \eta = 0, \tau = \alpha^2 e^{\frac{t}{\alpha}}$ and $\phi = -\alpha^2 u e^{\frac{t}{\alpha}}$. Therefore,

$$\begin{pmatrix} D_t\tau & D_x\tau & D_y\tau \\ D_t\xi & D_x\xi & D_y\xi \\ D_t\eta & D_x\eta & D_y\eta \end{pmatrix} \tag{3.12}$$

$$= \begin{pmatrix} \alpha e^{\frac{t}{\alpha}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.13}$$

and

$$(D_t\tau + D_x\xi + D_y\eta) \begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} = \alpha e^{\frac{t}{\alpha}} \begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix}. \tag{3.14}$$

Substituting Eqs.(3.11, 3.13), and (3.14) into Eq.(3.9), we conclude that X_4 is associated with $T^1 = (T^t, T^x, T^y)$ with $(Q_1 = -\frac{e^t}{x})$. Thus, we can get a reduced conserved vector by X_4 where X_4 has a canonical form $Y_4 = \frac{\partial}{\partial q}$ when

$$\frac{e^{\frac{-t}{\alpha}} dt}{\alpha^2} = \frac{dx}{0} = \frac{dy}{0} = \frac{e^{\frac{-t}{\alpha}} du}{-\alpha^2 u} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dw}{0}. \tag{3.15}$$

The invariants of X_4 from Eq.(3.15) are given by

$$b_1 = x,\ b_2 = y,\ b_3 = e^t u,\ b_4 = r,\ b_5 = s,\ b_6 + q = -\frac{1}{\alpha} e^{\frac{-t}{\alpha}},\ b_7 = w, \tag{3.16}$$

where b_4, b_5, b_6 and b_7 are arbitrary functions all dependent on b_1, b_2, and b_3.

By choosing $b_7 = b_3$, $b_5 = b_2$, $b_6 = 0$ and $b_4 = b_1$, we obtain the canonical coordinates

$$r = x,\ s = y,\ q = -\frac{1}{\alpha}e^{\frac{-t}{\alpha}},\ w = e^t u. \tag{3.17}$$

where $w = w(r, s)$, since $Y_4 = \frac{\partial}{\partial q}$.

From (3.17), the inverse canonical coordinates are given by

$$x = r,\ y = s,\ t = -\alpha \ln(-\alpha q),\ u = e^{-t}w. \tag{3.18}$$

In the light of the above similarity variables, the partial derivatives of u are given by

$$u_x = e^{-t}w_r,\ u_{xx} = e^{-t}w_{rr},\ u_y = e^{-t}w_s,\ u_{yy} = e^{-t}w_{ss}, u_t = -e^{-t}w. \tag{3.19}$$

Consequently, the Eq.(3.2) reduces to

$$r(\alpha w)^{\frac{1}{\alpha}}w_r + r^2(\alpha w)^{\frac{1}{\alpha}-1}w_r^2 + r^2(\alpha w)^{\frac{1}{\alpha}}w_{rr} + (\alpha w)^{\frac{1}{\alpha}-1}w_s^2 + (\alpha w)^{\frac{1}{\alpha}}w_{ss} = 0. \tag{3.20}$$

The inverse A^{-1} is given by

$$A^{-1} = \begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{\alpha^2}e^{\frac{-t}{\alpha}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.21}$$

In order to get the reduced conserved form, we use the following formula

$$\begin{pmatrix} T^r \\ \\ T^s \\ \\ T^q \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^t \\ \\ T^x \\ \\ T^y \end{pmatrix}. \tag{3.22}$$

We have $J = det(A) = \alpha^2 e^{\frac{t}{\alpha}}$, therefore,

$$\begin{pmatrix} T^r \\ \\ T^s \\ \\ T^q \end{pmatrix} = \alpha^2 e^{\frac{t}{\alpha}} \begin{pmatrix} 0 & 1 & 0 \\ \\ 0 & 0 & 1 \\ \\ \frac{1}{\alpha^2} e^{\frac{-t}{\alpha}} & 0 & 0 \end{pmatrix} \begin{pmatrix} T^t \\ \\ T^x \\ \\ T^y \end{pmatrix} \tag{3.23}$$

Thus, the reduced conserved form is

$$D_r T^r + D_s T^s = 0 \tag{3.24}$$

where

$$\begin{aligned} T^r &= -\alpha^2 r(\alpha w)^{\frac{1}{\alpha}} w_r, \\ T^s &= \frac{-\alpha^2 (\alpha w)^{\frac{1}{\alpha}}}{r} w_s, \\ T^q &= rw. \end{aligned} \tag{3.25}$$

The reduced conserved form admits the inherited symmetry

$$\widehat{X}_2 = \cos(s)\frac{\partial}{\partial r} - \frac{\sin(s)}{r}\frac{\partial}{\partial s}. \tag{3.26}$$

This symmetry is associated with the reduced conserved form. To show this symmetry is associated, from

$$\widehat{X}_2^{[1]} = \cos(s)\frac{\partial}{\partial r} - \frac{\sin(s)}{r}\frac{\partial}{\partial s} - \frac{\sin(s)}{r^2}w_s\frac{\partial}{\partial w_r} + \left(\frac{\cos(s)}{r}w_s + \sin(s)\,w_r\right)\frac{\partial}{\partial w_s}, \tag{3.27}$$

we get

$$\widehat{X}_2^{[1]}\begin{pmatrix} T^r \\ T^s \end{pmatrix} - \begin{pmatrix} D_r\xi^r & D_s\xi^r \\ D_r\xi^s & D_s\xi^s \end{pmatrix}\begin{pmatrix} T^r \\ T^s \end{pmatrix} + (D_r\xi^r + D_s\xi^s)\begin{pmatrix} T^r \\ T^s \end{pmatrix} = 0 \tag{3.28}$$

We transform $\widehat{\mathbf{X}}_2$ to its canonical form $Y = \frac{\partial}{\partial m}$. Thus, the canonical coordinates are

$$n = r\sin(s), \quad m = r(\cos(s) + \sin(s)), \quad v(n) = w. \tag{3.29}$$

Consequently, we have

$$A^{-1} = \begin{pmatrix} D_r n & D_r m \\ D_s n & D_s m \end{pmatrix} = \begin{pmatrix} \sin(s) & \cos(s) + \sin(s) \\ r\cos(s) & r(-\sin(s) + \cos(s)) \end{pmatrix} \tag{3.30}$$

In accordance with the similarities in Eq.(3.31), the partial derivatives of w are given by

$$w_r = \sin(s)v_n, \quad w_{rr} = \sin^2(s)v_{nn}, \quad w_s = r\cos(s)v_n, \quad w_{ss} = -r\sin(s)v_n + r^2\cos^2(s)v_{nn}. \tag{3.31}$$

Thus, the (3.20) reduces to

$$(\alpha v)v_{nn} + v_n^2 = 0. \tag{3.32}$$

To obtain the reduced conserved form, we apply the following formula

$$\begin{pmatrix} T^n \\ T^m \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^r \\ T^s \end{pmatrix}. \tag{3.33}$$

Thus,

$$\begin{pmatrix} T^n \\ T^m \end{pmatrix} = -\frac{1}{r} \begin{pmatrix} \sin(s) & r\cos(s) \\ \cos(s)+\sin(s) & r(\cos(s)-\sin(s)) \end{pmatrix} \begin{pmatrix} T^r \\ T^s \end{pmatrix} \tag{3.34}$$

where $J = det(A) = -\frac{1}{r}$. Using Eq.(3.31), we obtain

$$T^n = \alpha^2(\alpha v)^{\frac{1}{\alpha}} v_n. \tag{3.35}$$

where the reduced conserved form is given by

$$D_n T^n = 0. \tag{3.36}$$

According to Eqs.(3.35) and (3.36), the second step of the reduction is given by

$$\alpha^2(\alpha v)^{\frac{1}{\alpha}} v_n = C. \tag{3.37}$$

where C is a constant, $n = x \sin y$ and $v = e^t u$.

Choosing $\alpha = \frac{1}{2}$, with $f(x) = 1$ and $N^2 = 1$, the Eq. (3.2) becomes

$$\begin{aligned} &\frac{1}{4}x^2u^2u_{xx} + \frac{1}{2}x^2uu_x^2 + \frac{1}{4}xu^2u_x + \frac{1}{2}uu_y^2 \\ &+ \frac{1}{4}u^2u_{yy} - x^2u - x^2u_t = 0. \end{aligned} \tag{3.38}$$

In accordance with Eq. (3.37), Eq. (3.38) reduce to

$$\frac{1}{16}v^2 v_n = C_1. \tag{3.39}$$

Solving the Eq. (3.39) and using the backward substitution, then solution of Eq. (3.38) is

$$u(x, y, t) = \sqrt[3]{48C_1 x \exp(-3t) \sin y + C_2 \exp(-3t)}. \tag{3.40}$$

3.3 Conclusion

In this chapter, we have used the multiplier approach to obtain conservation laws for the $(2+1)-$dimensional cylindrical fin equation. Using the Lie symmetries associated with the resulting conserved vector to achieve double reduction. For a particular case of f and k, an exact solution has been found.

CHAPTER 4

A Symmetry Classification, Reductions and Exact Solutions of Non-linear (2+1) Spherical Fin Equation

4.1 Introduction

As we noted in the [Ch.1], the fin with spherical shapes also arise in applications. In this chapter, we present a complete classification of the nonlinear fin equation in spherical coordinates. The governing equation in this case is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k(u)\frac{\partial u}{\partial r}\right)+\frac{1}{r sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r}sin\theta k(u)\frac{\partial u}{\partial\theta}\right)\right]-N^2 f(r)u=u_t \tag{4.1}$$

The above equation can be rewritten as

$$r^2k(u)u_{rr}+r^2k_u u_r^2+2rk(u)u_r+k(u)u_{\theta\theta}+k_u u_\theta^2+\frac{\cos\theta}{\sin\theta}k(u)u_\theta-r^2N^2f(r)u-r^2u_t=0 \tag{4.2}$$

Using the substitution $r = x$, and $\cos\theta = y$, the above equation is transformed to

$$\begin{aligned} &x^2k(u)u_{xx} + x^2k_u u_x^2 + 2xk(u)u_x + k(u)(1-y^2)u_{yy} + k_u(1-y^2)u_y^2 \\ &- 2yk(u)u_y - x^2N^2f(x)u - x^2u_t = 0 \end{aligned} \tag{4.3}$$

In what follows, we use the well known Lie symmetry method [6, 24, 27, 43] to provide a complete classification of the above equation. In addition, reductions using two dimensional Lie subalgebras of the equation, to first or second order ordinary differential equations are given. The exact solution of a few interesting cases are obtained.

4.2 The Analysis of Fin Equation using Lie Point Symmetry Method

In this section, we perform the symmetry analysis of the Eq.(4.3). To this end, we use the Lie symmetry method described earlier in [Ch.2]. The symmetry generator associated with Eq.(4.3) is given by

$$X = \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \phi(x,y,t,u)\frac{\partial}{\partial u},$$

Requiring invariance of Eq.(4.3) with respect to the prolonged symmetry generator yields,

$$
\begin{aligned}
X^{(2)} = X + \phi^{x}\frac{\partial}{\partial u_x} + \phi^{y}\frac{\partial}{\partial u_y} + \phi^{t}\frac{\partial}{\partial u_t} + \phi^{xy}\frac{\partial}{\partial u_{xy}} + \\
\phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{yt}\frac{\partial}{\partial u_{yt}} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{yy}\frac{\partial}{\partial u_{yy}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}.
\end{aligned}
\tag{4.4}
$$

In the above expression, the coefficients of the prolonged generator are functions of (x, y, t, u) and can be determined by the formulae

$$\phi^{i} = D_i(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,i} + \tau u_{t,i}\,,$$

$$\phi^{ij} = D_i D_j(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij},$$

where D_i represents total derivative and subscripts of u partial derivative with respect to the respective coordinates. At this stage we use the Lie symmetry criterion that the PDE (4.3) is invariant under the prolonged symmetry generator (4.4) modulu the PDE, namely,

$$X^{(2)}[x^2 k(u) u_{xx} + x^2 k(u)_u u_x^2 + x k(u) u_x + k(u)_u u_y^2 + k(u) u_{yy} - x^2 N^2 f(x) u - x^2 u_t]\big|_{PDE(4.3)} = 0. \tag{4.5}$$

i.e. whenever $u_t = \frac{1}{x^2}[x^2 k(u) u_{xx} + x^2 k_u u_x^2 + 2x k(u) u_x + k(u)(1-y^2) u_{yy} + k_u (1-y^2) u_y^2$ $- 2y k(u) u_y - x^2 N^2 f(x) u]$.

Using results from Eq.(4.5) and comparing terms involving derivatives of the dependent function u, leads to the following over determined system of linear PDEs in ξ, η, τ and ϕ:

$$\xi_u = 0 = \eta_u = \tau_u = \tau_x = \tau_y = \phi_{uu}, \tag{4.6}$$

$$k_u\phi - 2k\xi_x + k\tau_t = 0, \tag{4.7}$$

$$x^2\eta_x + (1-y^2)\xi_y = 0, \tag{4.8}$$

$$\begin{aligned}&-2k\xi + 2xk_u\phi + x^2\xi_t - 2kx\xi_x + 2ky\xi_y + 2xk\tau_t \\ &-kx^2\xi_{xx} - k\xi_{yy} + ky^2\xi_{yy} + 2kx^2\phi_{xu} = 0,\end{aligned} \tag{4.9}$$

$$\begin{aligned}&-2xyk\eta - 2k\xi + 2ky^2\xi + x\phi k_u - xy^2\phi k_u \\ &-2xk\eta_y + 2xky^2\eta_y + xk\tau_t - xky^2\tau_t = 0,\end{aligned} \tag{4.10}$$

$$\begin{aligned}&-2xk\eta + 4ky\xi - 2xyk_u\phi + x^3\eta_t - 2kx^2\eta_x + 2xyk\eta_y - 2xyk\tau_t \\ &-x^3k\eta_{xx} - xk\eta_{yy} + xy^2k\eta_{yy} + 2xk\phi_{yu} - 2xy^2k\phi_{yu} = 0,\end{aligned} \tag{4.11}$$

$$\begin{aligned}&-N^2x^2f_xu\xi - N^2x^2f\phi + 2xk\phi_x - x^2\phi_t + x^2k\phi_{xx} + k\phi_{yy} \\ &-y^2k\phi_{yy} + N^2x^2fu\phi_u - N^2x^2fu\phi_u - n^2x^2fu\tau_t - 2yk\phi_y = 0.\end{aligned} \tag{4.12}$$

To determine the unknown functions ξ, τ, η and ϕ, we solve the above system starting by first considering Eq.(4.7), we have

$$\phi = \frac{k}{k_u}(2\xi_x - \tau_t) \tag{4.13}$$

Differentiating Eq.(4.7) w.r.t u twice yields

$$\phi_{uu} = (\frac{k}{k_u})_{uu}(2\xi_x - \tau_t) \tag{4.14}$$

Using Eq.(4.6) in Eq.(4.14) leads to

$$(\frac{k}{k_u})_{uu}(2\xi_x - \tau_t) = 0. \tag{4.15}$$

In what follows, we consider the above equation to perform a complete classification of both k and f.

4.3 Classification

In this section, we provide a complete classification of solutions of Eq.(4.3). Firstly, we notice that the following three cases arise from Eq.(4.15):

(I) $(\frac{k}{k_u})_{uu} = 0$,

(II) $2\xi_x - \tau_t = 0$,

(III) $2\xi_x - \tau_t = 0 = (\frac{k}{k_u})_{uu}$.

For complete classification, we consider all the three cases one by one.

4.1 Case I

Solving the differential Eq. $(\frac{k}{k_u})_{uu} = 0$, we determine $k(u)$ as,

$$k(u) = \gamma(\alpha u + \beta)^{\frac{1}{\alpha}}, \tag{4.16}$$

where γ, α and β are some integration constants. Using (4.16) into Eq.(4.13), instantly gives

$$\phi = (\alpha u + \beta)(2\xi_x - \tau_t). \tag{4.17}$$

Using Eq.(4.16) and Eq.(4.17) into Eq.(4.9), yields

$$\begin{aligned}&-2\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi+2x\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi_x+x^2\xi_t+2y\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi_y\\&+(4\alpha-1)x^2\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi_{xx}-(1-y^2)\gamma(\alpha u+\beta)^{\frac{1}{\alpha}}\xi_{yy}=0.\end{aligned} \tag{4.18}$$

Differentiating Eq.(4.18) w.r.t u gives

$$-2\gamma(\alpha u+\beta)^{\frac{1}{\alpha}-1}\left[\xi-x\xi_x-y\xi_y-(4\alpha-1)x^2\xi_{xx}+(1-y^2)\xi_{yy}\right]=0. \tag{4.19}$$

All constants involved in the above Eqs. are non-zero. Thus this is satisfied only when $\xi-x\xi_x-y\xi_y-(4\alpha-1)x^2\xi_{xx}+(1-y^2)\xi_{yy}=0$ (the case $\alpha=1$ not be considered as it becomes a special case of (I.b) that is dealt with later. The Ansatz solution of the above equation is

$$\xi=\lambda_1(t)x+\lambda_2(t)y. \tag{4.20}$$

Using Eq.(4.20) into Eq.(4.18) yields

$$\xi=c_1x+c_2y. \tag{4.21}$$

Using Eq.(4.21) into Eq.(4.8) yields

$$\eta=\tfrac{1-y^2}{x}c_2+\gamma(y,t). \tag{4.22}$$

To determine $\gamma(y,t)$ we use Eq.(4.22) into Eq.(4.10) to find that,

$$\gamma(y,t) = \sqrt{1-y^2}\beta(t). \tag{4.23}$$

Therefore, Eq.(4.22) becomes

$$\eta = \tfrac{1-y^2}{x}c_2 + \sqrt{1-y^2}\delta(t). \tag{4.24}$$

Again, to determine $\delta(t)$, we use Eq.(4.24) into Eq.(4.11), to infer that,

$$\left[-2kx\sqrt{1-y^2} - \tfrac{2kxy^2}{\sqrt{1-y^2}} + kx(1-y^2)^{\frac{-3}{2}} - kxy^2(1-y^2)^{\frac{-3}{2}}\right]\delta(t) + x^3\sqrt{1-y^2}\delta_t(t) = 0. \tag{4.25}$$

The solution of the above equation is the trivial solution which is $\delta(t) = 0$. Consequently, the Eq.(4.24) becomes

$$\eta = \tfrac{1-y^2}{x}c_2. \tag{4.26}$$

Using Eq.(4.21) and Eq.(4.26) into (4.12) yields

$$-N^3x^3f_xuc_1 - N^2x^2yf_xuc_2 - \beta N^2x^2f(2c_1-\tau_t) + x^2(\alpha u+\beta)\tau_{tt} - N^2x^2fu\tau_t = 0. \tag{4.27}$$

Differentiating Eq.(4.27) with respect to t gives

$$\beta N^2x^2f\tau_{tt} + x^2(\alpha u+\beta)\tau_{ttt} - N^2x^2fu\tau_{tt} = 0. \tag{4.28}$$

Again, differentiating Eq.(4.28) w.r.t t , we obtain

$$\frac{\tau_{ttt}}{\tau_{tt}} = \frac{N^2 f}{\alpha}. \tag{4.29}$$

This implies that $f(x) = c$ and hence

$$\tau(t) = \frac{c_3\alpha^2}{N^4c^2}e^{\frac{N^2c}{\alpha}} + c_4 t + c_5. \tag{4.30}$$

Using Eq.(4.30) with $f(x) = c$ into (4.28), yields

$$\beta c c_3(1 + \tfrac{1}{\alpha}) = 0. \tag{4.31}$$

From Eq.(4.31) four cases arise:

(I.a) $\beta = 0,\ c_3 \neq 0\, and\ \alpha > 0,$

(I.b) $\beta \neq 0,\ c_3 = 0,\ and\ \alpha > 0,$

(I.c) $\beta \neq 0,\ c_3 \neq 0,\ and\ \alpha = -1,$

(I.d) $\beta = 0,\ c_3 \neq 0,\ and\ \alpha = -1.$

We first consider I.a.

4.1.1. Subcase (I.a.) $k(u) = \gamma(\alpha u)^{\frac{1}{\alpha}}$ and $f(x) = c.$

Using Eq.(4.30) into Eq.(4.27), leads to $c_4 = 0$. Therefore, the expression for the infinitesimal symmetry generators $\xi, \eta, \tau,$ and ϕ take the form,

$$\begin{aligned} &\xi = c_1 x + c_2 y,\ \eta = \frac{(1-y^2)}{x} c_2,\ \tau = \frac{c_3 \alpha}{c N^2} \exp(\frac{N^2 c}{\alpha} t) + c_5, \\ &\phi = \alpha u (2c_1 - c_3 c N^2 \exp(\frac{N^2 c}{\alpha} t)). \end{aligned} \tag{4.32}$$

The $four$ symmetry generators associated with above infinitesimals are given by,

$$\begin{aligned} &X_1 = x\frac{\partial}{\partial x} + 2\alpha u \frac{\partial}{\partial u}, \quad X_2 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \\ &X_3 = \frac{\alpha}{cN^2} e^{\frac{N^2 c}{\alpha} t}\frac{\partial}{\partial t} - \alpha u e^{\frac{N^2 c}{\alpha} t}\frac{\partial}{\partial u}, \quad X_4 = \frac{\partial}{\partial t}. \end{aligned} \tag{4.33}$$

4.1.2. Subcase (I.b.) $k(u) = \gamma(\alpha u + \beta)^{\frac{1}{\alpha}}$ and $f(x) = c$.

Using Eq.(4.30) into Eq.(4.27) with the above values of k and f , lead to $c_4 = 0 = c_1$. Therefore, the expression for the infinitesimal symmetry generators ξ, η, τ and ϕ take the form,

$$\xi = c_2 y, \quad \eta = \tfrac{(1-y^2)}{x} c_2, \quad \tau = c_5, \quad \phi = 0. \tag{4.34}$$

The two symmetry generators associated with above infinitesimals are given by,

$$X_1 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}. \tag{4.35}$$

4.1.3. Subcase (I.c.) $k(u) = \frac{\gamma}{\beta - u}$ and $f(x) = c$.

Using Eq.(4.30) into Eq.(4.27) with the above values of k and f, lead to $c_4 = 0 = c_1 = c_3$. Therefore, the expression for the infinitesimal symmetry generators

$\xi, \eta, \tau,$ and ϕ take the form,

$$\xi = c_2 y, \quad \eta = \tfrac{(1-y^2)}{x} c_2, \quad \tau = c_5, \quad \phi = 0. \tag{4.36}$$

The *two* symmetry generators associated with above infinitesimals are given by,

$$X_1 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}. \tag{4.37}$$

4.1.4. Subcase (I.d.) $k(u) = -\frac{\gamma}{u}$ and $f(x) = c$.

Using Eq.(4.30) into Eq.(4.27) with the above values of k and f , lead to $c_4 = 0$. Therefore, the expression for the infinitesimal symmetry generators $\xi, \eta, \tau,$ and ϕ take the form,

$$\begin{aligned} &\xi = c_1 x + c_2 y, \quad \eta = \tfrac{(1-y^2)}{x} c_2, \\ &\tau = \tfrac{-c_6}{cN^2} e^{-cN^2 t} + c_5, \quad \phi = -u(2c_1 - c_3 e^{-cN^2 t}). \end{aligned} \tag{4.38}$$

The $four$ symmetry generators associated with above infinitesimals are given by,

$$\begin{aligned} &X_1 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}, \quad X_2 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \\ &X_3 = -\frac{1}{cN^2}e^{-N^2 ct}\frac{\partial}{\partial t} + ue^{-N^2 ct}\frac{\partial}{\partial u}, \quad X_4 = \frac{\partial}{\partial t}. \end{aligned} \tag{4.39}$$

4.2 Case II

In accordance with $2\xi_x - \tau_t = 0$, the system ((4.6)-(4.12)) becomes

$$\xi_u = 0 = \eta_u = \tau_u = \phi = \tau_y = \tau_x, \tag{4.40}$$

$$x^2\eta_x + (1-y^2)\xi_y = 0, \tag{4.41}$$

$$-2k\xi + x^2\xi_t + 2kx\xi_x + 2ky\xi_y - kx^2\xi_{xx} - k\xi_{yy} + ky^2\xi_{yy} = 0, \tag{4.42}$$

$$-2xyk\eta - 2k\xi + 2ky^2\xi - 2xk\eta_y + 2xky^2\eta_y + 2xk\xi_x - 2xky^2\xi_x = 0, \tag{4.43}$$

$$\begin{aligned} &-2xk\eta + 4ky\xi - 2xyk_u\phi + x^3\eta_t - 2kx^2\eta_x + 2xyk\eta_y - 4xyk\xi_x \\ &-x^3k\eta_{xx} - xk\eta_{yy} + xy^2k\eta_{yy} = 0, \end{aligned} \tag{4.44}$$

$$-N^2x^2f_xu\xi - 2N^2x^2fu\xi_x = 0. \tag{4.45}$$

From Eq.(4.41), we have

$$\eta_x = -\tfrac{(1-y^2)}{x^2}\xi_y. \tag{4.46}$$

Differentiation Eq.(4.42) with respect to u yields

$$-2k_u\xi + 2k_ux\xi_x + 2k_uy\xi_y - k_ux^2\xi_{xx} - k_u\xi_{yy} + k_uy^2\xi_{yy} = 0, \tag{4.47}$$

then,

$$k_u\left(-2\xi + 2\xi_x + 2y\xi_y - x^2\xi_{xx} - \xi_{yy} + y^2\xi_{yy}\right) = 0. \tag{4.48}$$

This leads to

$$\left(-2\xi + 2\xi_x + 2y\xi_y - x^2\xi_{xx} - \xi_{yy} + y^2\xi_{yy}\right) = 0, \tag{4.49}$$

From previous case, the solution of (4.49) is given by

$$\xi = c_1 x + c_2 y. \tag{4.50}$$

We follow the same procedure followed in the previous case, we end up with the following expressions of ξ,,η τ and ϕ,namely,

$$\begin{aligned} &\xi = c_1 x + c_2 y, \quad \eta = \tfrac{(1-y^2)}{x} c_2, \\ &\tau = 2c_1, \quad \phi = 0. \end{aligned} \tag{4.51}$$

The above values of ξ,,η τ and ϕ satisfy the system ((4.40)-(4.44)). At this stage, we use (4.51) in (4.45) to get,

$$-f_x(c_1 x + c_2 y) - 2c_1 f = 0 \tag{4.52}$$

Differentiating Eq.(4.52) w.r.t y, we obtain

$$-c_2 f_x = 0. \tag{4.53}$$

From Eq.(4.53), two cases arise:

(II.a) $c_2 = 0, and\ f_x \neq 0,$

(II.b) $c_2 \neq 0, and\ f_x = 0.$

First, we consider (II.a).

4.2.1 Case II.a

Using theses conditions arising in this case into Eq.((4.52), gives

$$-c_1[xf_x + 2f] = 0 \tag{4.54}$$

From Eq.(4.54), two cases arise:

(II.a.1) $c_1 = 0, and\ xf_x + 2f \neq 0,$

(II.a.2) $c_1 \neq 0, and\ xf_x + 2f = 0.$

Considering first (II.a.1).

4.2.1.1 Case II.a.1

In the light of the conditions of this case, the $k(u)$, and $f(x)$ are arbitrary functions and the general expressions of ξ, η, tau, and ϕ, have the following form:

$$\xi = \eta = \phi = 0, \quad \tau = c_3. \tag{4.55}$$

the only one generator corresponding to this case is $\mathbf{X} = \frac{\partial}{\partial \mathbf{t}}$.

4.2.1.1 Case II.a.2

In accordance with these conditions of this case, the $k(u)$, is arbitrary function, $f(x) = \frac{c}{x^2}$ and the general expressions of ξ, η, tau, and ϕ, have the following form:

$$\xi = c_1 x, \quad \eta = 0, \quad \tau = 2c_1 t + c_3, \quad \phi = 0. \tag{4.56}$$

and the generators in this case are

$$X_1 = x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial t}. \tag{4.57}$$

4.2.2 Case II.b ($c_2 \neq 0$ and $f(x) = c$.)

Using theses conditions arising in this case into Eq.((4.52),gives $c_1 = 0$. Thus, we infer that $f(x) = c$ and $k(u)$ is arbitrary, and hence the general expression of ξ, η, τ, and ϕ are

$$\xi = c_2 y, \quad \eta = \tfrac{(1-y^2)}{x} c_2, \quad \tau = c_3, \quad \phi = 0. \tag{4.58}$$

The symmetry generators in this case are,

$$X_1 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}. \tag{4.59}$$

4.4 Symmetry Generators

In this section, we list the Lie symmetry generators obtained above for different values of $k(u)$ and $f(x)$.

1-$f(x) = c$

a- $k(u) = \gamma(\alpha u)^{\frac{1}{\alpha}}$. In this case the symmetry generators are

$$X_1 = x\frac{\partial}{\partial x} + 2\alpha u\frac{\partial}{\partial u}, \quad X_2 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial t},$$
$$X_4 = \frac{\alpha}{N^2 c}\exp(\frac{N^2 c}{\alpha}t)\frac{\partial}{\partial t} - \alpha u \exp(\frac{N^2 c}{\alpha}t)\frac{\partial}{\partial u}.$$

The commutation relation for these generators are given in the following table

Table 4.1: Commutator table of the fin equation

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	$-X_2$	0	X_2
X_2	X_2	0	0	X_1
X_3	0	0	0	$\frac{cN^2}{\alpha}X_4$
X_4	0	0	$-\frac{cN^2}{\alpha}X_4$	0

b- $k(u) = \gamma(\alpha u + \beta)^{\frac{1}{\alpha}}$. In this case the symmetry generators are

$$X_1 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}.$$

The commutation relation for these generators are given in the following table

Table 4.2: Commutator table of the fin equation

$[X_i, X_j]$	X_1	X_2
X_1	0	0
X_2	0	0

c- $k(u) = \frac{\gamma}{(\beta-u)}$. In this case the symmetry generators are

$$X_1 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}.$$

The commutation relation for these generators are given in the following table

Table 4.3: Commutator table of the fin equation

$[X_i, X_j]$	X_1	X_2
X_1	0	0
X_2	0	0

d- $k(u) = -\frac{\gamma}{u}$. In this case the symmetry generators are

$$X_1 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}, \quad X_2 = y\frac{\partial}{\partial x} + \frac{(1-y^2)}{x}\frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial t},$$
$$X_4 = -\frac{1}{N^2 c}\exp(-N^2 ct)\frac{\partial}{\partial t} + u\exp(-N^2 ct)\frac{\partial}{\partial u}.$$

The commutation relation for these generators are given in the following table

Table 4.4: Commutator table of the fin equation

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	$-X_2$	0	X_2
X_2	X_2	0	0	X_1
X_3	0	0	0	$-cN^2X_4$
X_4	0	0	cN^2X_4	0

2- $f(x)$ and $k(u)$ are arbitrary functions.

In this case, we have only one generator which is $X = \frac{\partial}{\partial t}$.

3- $f(x) = \dfrac{c}{x^2}$ and $k(u)$is arbitrary.

In this case, we have only two generators which are

$$X_1 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial t}.$$

The commutation relation for these generators are given in the following table

Table 4.5: Commutator table of the fin equation

$[X_i, X_j]$	X_1	X_2
X_1	0	$-2X_2$
X_2	$2X_2$	0

4.5 Reduction under two dimensional subalgebra

In what follows, we will show the reduction of the given problem to an ODE using two dimensional subalgebra.

4.5.1 Case 1

In this subsection, we present solutions of Eq.(4.3) via reductions. These reductions are obtained by the similarity variables obtained through symmetry generators. To perform reductions of Eq.(4.3), we first consider two symmetry generators, from Table (4.1) (In this case $k(u) = \gamma(\alpha u)^{\frac{1}{\alpha}}$ and $f(x) = c$). Here X_1, and X_3 span an abelian subalgebra. To start reduction, we first consider X_3. The characteristic equation corresponding to this generator,

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}. \tag{4.60}$$

Solving the above equation it is straight forward [5] to find that it yields the similarity variables, $r = x$ and $s = y$ with $w(r, s) = u$. Replacing u in Eq.(4.3) in terms of new variables becomes,

$$
\begin{aligned}
&r^2k(w)w_{rr} + r^2k_w w_r^2 + 2rk(w)w_r + k(w)(1-s^2)w_{ss} \\
&+ (1-s^2)k_w w_s^2 - 2sk(w)w_s - N^2cr^2w = 0.
\end{aligned}
\tag{4.61}
$$

To proceed further, we first transform X_1 in terms of new variables r, s and w. Thus, $\widehat{X}_1 = r\frac{\partial}{\partial r} + 2\alpha w\frac{\partial}{\partial w}$. The similarities corresponding to this generator are $z = s$ and $v(z) = r^{2\alpha}w$. This reduces Eq.(4.61) to a second-order differential equation given by,

$$
\begin{aligned}
&2\gamma\alpha^{\frac{1}{\alpha}+1}(2\alpha-1)v^{\frac{1}{\alpha}+1} + 8\gamma\alpha^{\frac{1}{\alpha}+1}v^{\frac{1}{\alpha}+1} + \gamma\alpha^{\frac{1}{\alpha}}(1-z^2)v^{\frac{1}{\alpha}}v_{zz} \\
&+ \gamma\alpha^{\frac{1}{\alpha}-1}(1-z^2)v^{\frac{1}{\alpha}-1}v_z^2 - 2\gamma\alpha^{\frac{1}{\alpha}}zv^{\frac{1}{\alpha}}v_z - N^2cv = 0.
\end{aligned}
\tag{4.62}
$$

4.5.2 Case 2

From Table (4.1), $[X_1, X_4] = 0$ are commutative. Thus, the reduction can be started either by X_1 or X_4. To this end, we first consider X_1. The characteristic equation corresponding to this generator is

$$
\frac{dx}{x} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{2\alpha u} \tag{4.63}
$$

The similarity variables corresponding to above equation become $r = y$, $s = t$ and $u = x^{2\alpha} w$. These variables reduce Eq.(4.3) to a PDE of the form,

$$
\begin{aligned}
&2\gamma\alpha^{\frac{1}{\alpha}+1}(2\alpha - 1)w^{\frac{1}{\alpha}+1} + 8\gamma\alpha^{\frac{1}{\alpha}+1}w^{\frac{1}{\alpha}+1} + \gamma\alpha^{\frac{1}{\alpha}}(1 - r^2)w^{\frac{1}{\alpha}}w_{rr} \\
&+ \gamma\alpha^{\frac{1}{\alpha}-1}(1 - r^2)w^{\frac{1}{\alpha}-1}w_r^2 - 2\gamma\alpha^{\frac{1}{\alpha}} r w^{\frac{1}{\alpha}} w_r - N^2 c w - w_s = 0.
\end{aligned} \tag{4.64}
$$

Using similarity variables transformation obtained from X_4, transforms $\widehat{X}_4 = \frac{\alpha}{N^2 c}\exp(\frac{N^2 c}{\alpha}s)\frac{\partial}{\partial s} - \alpha w \exp(\frac{N^2 c}{\alpha}s)\frac{\partial}{\partial w}$. This leads to the new coordinates $r = z$, $v(z) = \exp(-N^2 c s)w$. In the light of these similarities, Eq.(4.64) transforms to,

$$
(4\alpha^2 - 6\alpha)v + (1 - z^2)v_{zz} + \frac{1}{\alpha}(1 - z^2)\frac{1}{v}v_z^2 - 2zv_z = 0. \tag{4.65}
$$

Choosing $\alpha = \frac{3}{2}$, the above equation takes the form,

$$
(1 - z^2)v_{zz} + \frac{2}{3}(1 - z^2)\frac{1}{v}v_z^2 - 2zv_z = 0. \tag{4.66}
$$

giving exact solution

$$
u(x, y, t) = c_2 x^3 \exp(-N^2 ct)\left(6c_1 + 5\ln(y - 1) - 5\ln(y + 1)\right)^{\frac{3}{5}}. \tag{4.67}
$$

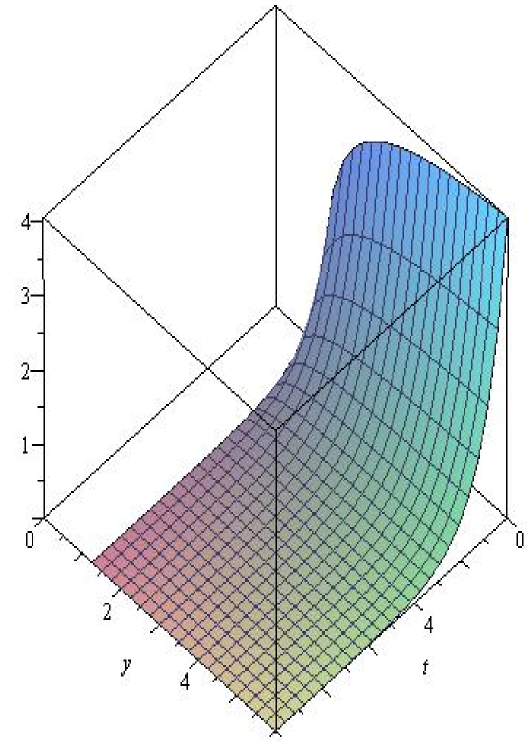

Figure 4.1: Plot of solution given by Eq. (4.67) with $c_1 = 2,\ N^2c = 1,\ c_2 = 1$ and $x = constant$.

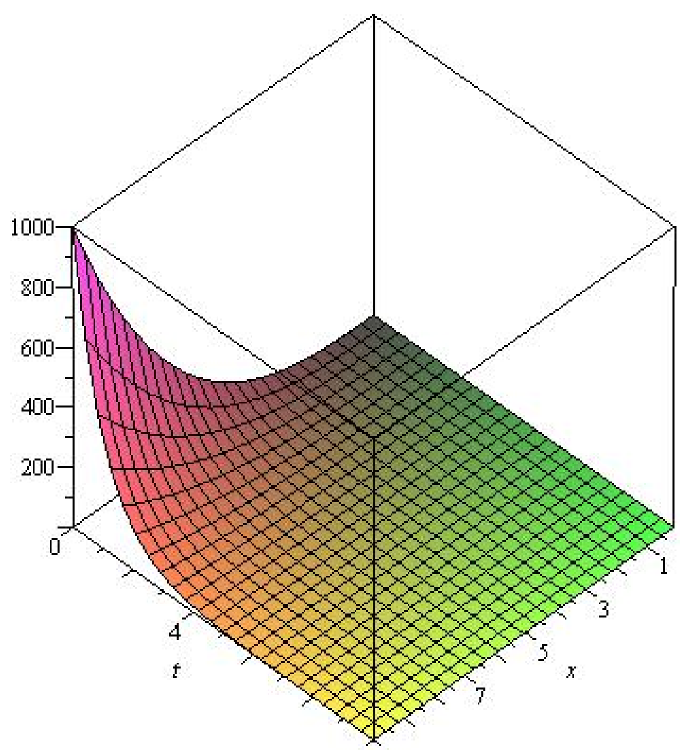

Figure 4.2: Plot of solution given by Eq. (4.67) with $c_1 = 0,\ N^2c = 1\ c_2 = 1$, and $y = constant$.

4.5.3 Case 3

In this case we execute reduction using table (4.5). Here $f(x) = \dfrac{c}{x^2}$ and $k(u)$ is arbitrary. We consider the symmetry generators X_1, X_2 which satisfy a commutative relationship $[X_1, X_2] = -2X_2$ as shown in table(4.5). First considering X_2, and follow the procedure in the previous cases, the generator X_2 reduces Eq.(4.3) to

$$\begin{aligned} &r^2k(w)w_{rr} + r^2k_w w_r^2 + 2rk(w)w_r + k(w)(1-s^2)w_{ss} \\ &+ (1-s^2)k_w w_s^2 - 2sk(w)w_s - N^2cw = 0. \end{aligned} \tag{4.68}$$

In the light of X_2, the X_1 transforms to $\widehat{X}_1 = r\frac{\partial}{\partial r}$ which gives $z = s$ with $w = v(z)$. In the light of these similarity variables, Eq.(4.68) reduces to the following ODE:

$$k(v)(1-z^2)v_{zz} + (1-z^2)k_v v_z^2 - 2zk(v)v_z - N^2cv = 0. \tag{4.69}$$

Remark: The reductions performed above are given in the tabular form in the following:

Case#	Algebra	Reduction	z	v
Case 1	$[X_1, X_3] = 0$	$2\gamma\alpha^{\frac{1}{\alpha}+1}(2\alpha-1)v^{\frac{1}{\alpha}+1} + 8\gamma\alpha^{\frac{1}{\alpha}+1}v^{\frac{1}{\alpha}+1} + \gamma\alpha^{\frac{1}{\alpha}}(1-z^2)v^{\frac{1}{\alpha}}v_{zz}$ $+\gamma\alpha^{\frac{1}{\alpha}-1}(1-z^2)v^{\frac{1}{\alpha}-1}v_z^2 - 2\gamma\alpha^{\frac{1}{\alpha}}zv^{\frac{1}{\alpha}}v_z - N^2cv = 0$	y	$x^{2\alpha}u$
Case 2	$[X_1, X_4] = 0$	$(4\alpha^2 - 6\alpha)v + (1-z^2)v_{zz} + \frac{1}{\alpha}(1-z^2)\frac{1}{v}v_z^2 - 2zv_z = 0$	y	$e^{-N^2ct}x^{2\alpha}u$
Case 3	$[X_1, X_2] = -2X_2$	$k(v)(1-z^2)v_{zz} + (1-z^2)k_v v_z^2 - 2zk(v)v_z - N^2cv = 0$	y	u

Table 4.6: Reduction

4.6 Conclusion

As a consequence of the results obtaining in this chapter, we notice that the reduction of the given equation to ODE may lead to find its exact solution. Some of these ODEs can not be solved readily. However, the reduced form is generally simpler than the original non-linear PDE and we may use symmetry or other methods to solve them.

CHAPTER 5

Conclusion and Future Work

In this dissertation we performed the Lie symmetry analysis to classify, reduce and find invariant solutions of a type of evolution equation arising in industrial application. This equation, known as the fin equation, has been studied in cylindrical and spherical coordinates to model the fins of these shapes respectively.

In chapter 3, we classified the nonlinear (2+1)-dimensional fin equation by considering cylindrical fins with a temperature dependent thermal conductivity and variable heat transfer coefficient. We obtained reductions and invariant exact solutions in accordance with this classification.

In chapter 4, we used the multiplier approach to find the conserved vectors that lead to the reduction of the given cylindrical fin equation accordingly to the principle of double reduction. Finally, we considered the nonlinear (2+1)-dimensional fin equation in spherical coordinates and obtained the classification of the thermal conductivity and the heat transfer coefficient. Then, we obtained the reduction and exact solutions in some interesting cases. We conclude that the study of the nonlinear (2+1)-dimensional fin equation in cylindrical and spherical coordinates via Lie symmetry analysis leads to invariant solutions for fins of different shapes. We also obtained the conservation law through multiplier approach to yield first integral and reduction of the nonlinear PDE for cylindrical fins.

The present study opens many possible future directions.

- The Lie symmetry generators obtained in the two cases of cylindrical and spherical coordinates can be further exploited to give more exact solutions.

- The non-isotropic fins, having thermal conductivity not constant may be considered. The governing equation will result in the thermal conductivity to be a function of not only the temperature but of position also.

- The (3+1)-dimensional fin equation can be considered from Lie-symmetry point of view. We anticipate that the analysis will be much more complicated.

- The non-homogeneous fin equation in which a source/sink will also be an interesting future study.

APPENDICES

Appendix A

Consider the following equation:

$$u_{tt} - u_{xx} - u = 0 \tag{A1}$$

In order to construct the conserved vector of Eq. (A1), we find, the multiplier Q, by applying the Euler-Lagrange operator $\dfrac{\delta}{\delta u}$, on Eq. (A1):

$$\frac{\delta}{\delta u}(Q(u_{tt} - u_{xx} - u)) = 0 \tag{A2}$$

Taking $Q = f(x, t, u, u_x, u_t, u_{xx}, u_{xt})$, the Euler-Lagrange operator takes the form:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_xD_x\frac{\partial}{\partial u_{xx}}D_xD_t\frac{\partial}{\partial u_{xt}} + D_tD_t\frac{\partial}{\partial u_{tt}}.$$

where $D_i, (i = x, t)$ represent the total derivative:

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + u_{txx}\frac{\partial}{\partial u_{xx}} + u_{txt}\frac{\partial}{\partial u_{xt}} + u_{ttt}\frac{\partial}{\partial u_{tt}}.$$
$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + u_{xxx}\frac{\partial}{\partial u_{xx}} + u_{xtx}\frac{\partial}{\partial u_{xt}} + u_{xtt}\frac{\partial}{\partial u_{tt}}.$$

In the light of above, Eq. (A2) becomes:

$$f_u(u_{tt} - u_{xx} - u) - f - f_{tu_t}(u_{tt} - u_{xx} - u) - u_t[f_{uu_t}(u_{tt} - u_{xx} - u) - f_{u_t}]$$
$$- u_{tt}[f_{u_tu_t}(u_{tt} - u_{xx} - u)] - u_{xt}[f_{u_tu_x}(u_{tt} - u_{xx} - u)]$$

$$- u_{txx}[f_{u_t u_{xx}}(u_{tt} - u_{xx} - u) - f_{u_t}] - u_{ttt} f_{u_t} - u_{ttx} f_{u_t u_{tx}}$$

$$(u_{tt} - u_{xx} - u) - f_{xu_x}(u_{tt} - u_{xx} - u) - u_x[f_{uu_x}(u_{tt} - u_{xx} - u) - f_{u_x}]$$

$$- u_{xx}[f_{u_x u_x}(u_{tt} - u_{xx} - u)] - u_{xt}[f_{u_t u_x}(u_{tt} - u_{xx} - u)] - u_{xxx}$$

$$[f_{u_{xx}} u_x(u_{tt} - u_{xx} - u) - f_{u_x}] - u_{xtt} f_{u_x} - u_{xxt} f_{u_{tx} u u_x}$$

$$(u_{tt} - u_{xx} - u) + f_{txu_{tx}}(u_{tt} - u_{xx} - u) + u_x[f_{tuu_{tx}}(u_{tt} - u_{xx} - u) - f_{tu_{tx}}]$$

$$+ u_{xx}[f_{txu_x u_{tx}}(u_{tt} - u_{xx} - u) - f_{tu_{tx}}] + u_{xt}[f_{txu_t u_{tx}}(u_{tt} - u_{xx} - u)]$$

$$+ u_{xxx}[f_{tu_{xx} u_{tx}}(u_{tt} - u_{xx} - u) - f_{tu_{tx}}] + u_{xtt} f_{tu_{tx}} + u_{txx} f_{tu_{tx} u_{tx}}$$

$$(u_{tt} - u_{xx} - u) + u_t\{f_{uxu_{tx}}(u_{tt} - u_{xx} - u) - f_{xu_{tx}} + u_x[f_{uuu_{tx}}(u_{tt} - u_{xx} - u)$$

$$- 2f_{uu_{tx}}] + u_{xx}[f_{uxu_x u_{tx}}(u_{tt} - u_{xx} - u) - f_{xu_x u_{tx}}] + u_{xt}[f_{uxu_t u_{tx}}$$

$$(u_{tt} - u_{xx} - u) - f_{xu_t u_{tx}}] + u_{xxx}[f_{uu_{xx} u_{tx}}(u_{tt} - u_{xx} - u) - f_{u_{xx} u_{tx}} - f_{uu_{tx}}]$$

$$+ u_{xtt} f_{uu_{tx}} - f_{u_{tx} u_{tx}} + u_{txx} f_{uu_{tx} u_{tx}}(u_{tt} - u_{xx} - u)\} + u_{tt}\{f_{xu_t u_{tx}}$$

$$(u_{tt} - u_{xx} - u) + u_x[f_{uu_t u_{tx}}(u_{tt} - u_{xx} - u) - f_{u_t u_{tx}}] + u_{xx}[f_{xu_x u_t u_{tx}}$$

$$(u_{tt} - u_{xx} - u) + u_{xt}[f_{xu_t u_x u_{tx}}(u_{tt} - u_{xx} - u) + u_{xt}[f_{xu_t u_t u_{xt}}(u_{tt} - u_{xx} - u)]$$

$$+ u_{xxx}[f_{u_t u_{xx} u_{tx}}(u_{tt} - u_{xx} - u) - f_{u_t u_{tx}}] + u_{xtt} f_{u_t u_{tx}} + u_{txx} f_{u_t u_{tx} u_{tx}}$$

$$(u_{tt} - u_{xx} - u)\} + u_{xt}\{f_x u_x u_{xt}(u_{tt} - u_{xx} - u) + u_x[f_{uu_x u_{tx}}(u_{tt} - u_{xx} - u) - f_{u_x u_{tx}}]$$

$$+ [f_{uu_{tx}}(u_{tt} - u_{xx} - u) - f_{u_{tx}}] + u_{xx}[f_{xu_x u_x u_{tx}}(u_{tt} - u_{xx} - u)] + u_{xt}[f_{xu_t u_x u_{tx}}$$

$$(u_{tt} - u_{xx} - u)] + u_{xxx}[f_{u_x u_{xx} u_{xt}}(u_{tt} - u_{xx} - u) - f_{u_x u_{tx}}] + u_{xtt} f_{u_x u_{tx}}$$

$$+ u_{txx} f_{u_x u_{tx} u_{tx}}(u_{tt} - u_{xx} - u)\} + u_{txx}\{f_{xu_{xx} u_{tx}}(u_{tt} - u_{xx} - u) - f_{xu_{tx}}$$

$$+ u_x[f_{uu_{xx} u_{tx}}(u_{tt} - u_{xx} - u) - f_{uu_{tx}} - f_{u_{xx} u_{tx}}] + [f_{xu_x u_{tx}}(u_{tt} - u_{xx} - u)]$$

$$+ u_{xx}[f_{xu_x u_{xx} u_{tx}}(u_{tt} - u_{xx} - u) - f_{xu_x u_{tx}}] + u_{xt}[f_{xu_t u_{xx} u_{tx}}$$

$$(u_{tt}-u_{xx}-u)-f_{xu_tu_{tx}}]+u_{xxx}[f_{u_{xx}u_{xx}u_{tx}}(u_{tt}-u_{xx}-u)-2f_{u_{xx}u_{tx}}]$$

$$+u_{xtt}f_{u_{xx}u_{tx}}+u_{txx}f_{u_{xx}u_{tx}u_{tx}}(u_{tt}-u_{xx}-u)-u_{txx}f_{u_{tx}u_{tx}}\}$$

$$+u_{ttt}\{f_{xu_{tx}}+u_xf_{uu_{tx}}+u_{xx}f_{xu_xu_{tx}}+u_{xt}f_{xu_tu_{tx}}$$

$$+u_{xxx}f_{u_{xx}u_{tx}}+u_{txx}f_{u_{tx}u_{tx}}\}+u_{ttx}\{f_{xu_{tx}u_{tx}}(u_{tt}-u_{xx}-u)$$

$$+u_x[f_{uu_{tx}u_{tx}}(u_{tt}-u_{xx}-u)-f_{u_{tx}u_{tx}}]+u_{xx}[f_{xu_xu_{tx}u_{tx}}(u_{tt}-u_{xx}-u)]$$

$$+[f_{xu_tu_{tx}}(u_{tt}-u_{xx}-u)]+u_{xt}[f_{xu_tu_{tx}u_{tx}}(u_{tt}-u_{xx}-u)]$$

$$+u_{xxx}[f_{u_{xx}u_{tx}u_{tx}}(u_{tt}-u_{xx}-u)-f_{u_{tx}u_{tx}}]+u_{xtt}f_{u_{tx}u_{tx}}$$

$$+u_{txx}f_{u_{tx}u_{tx}u_{tx}}(u_{tt}-u_{xx}-u)\}-f_{xx}+f_{xxu_{xx}}(u_{tt}-u_{xx}-u)$$

$$+u_x[-f_{xu}+f_{xuu_{xxx}}(u_{tt}-u_{xx}-u)-f_{xu_{xx}}]+u_{xx}[-f_{xu_x}+f_{xu_xu_{xx}}(u_{tt}-u_{xx}-u)]$$

$$+u_{xt}[-f_{xu_t}+f_{xu_tu_{xx}}(u_{tt}-u_{xx}-u)]+u_{xxx}[-f_{xu_{xx}}+f_{xu_{xx}u_{xx}}(u_{tt}-u_{xx}-u)$$

$$-f_{xu_{xx}}]+u_{xxt}f_{xu_{xx}}+u_{txx}[-f_{xu_{tx}}+f_{xu_{tx}u_{xx}}(u_{tt}-u_{xx}-u)]$$

$$+u_x\{-f_{xu}+f_{xuu_{xx}}(u_{tt}-u_{xx}-u)-f_{xuu_{xx}}+u_x[-f_{uu}+f_{uuu_{xx}}(u_{tt}-u_{xx}-u)-2f_{uu_{xx}}]$$

$$+u_{xx}[-f_{uu_x}+f_{uu_xu_{xx}}(u_{tt}-u_{xx}-u)-f_{u_{xx}u_{xx}}-f_{uu_{xx}}]+u_{xtt}f_{uu_{xx}}$$

$$+u_{txx}[-f_{uu_{tx}}+f_{uu_{tx}u}u_{tx}(u_{tt}-u_{xx}-u)-f_{u_{tx}u_{xx}}]+u_{xx}\{-f_{xu_x}+f_{xu_xu_{xx}}$$

$$(u_{tt}-u_{xx}-u)+u_x[-f_{uu_x}+f_{uu_xu_{xx}}(u_{tt}-u_{xx}-u)-f_{u_x}u_{xx}]+[-f_u$$

$$+f_{uu_{xx}}(u_{tt}-u_{xx}-u)-f_{u_{xx}}]+u_{xx}[-f_{u_xu_x}+f_{u_xu_xu_{xx}}(u_{tt}-u_{xx}-u)]$$

$$+u_{xt}[-f_{u_xu_t}+f_{u_xu_tu_{xx}}(u_{tt}-u_{xx}-u)]+u_{xxx}[-f_{u_xu_{xx}}$$

$$+f_{u_xu_{xx}u_{xx}}(u_{tt}-u_{xx}-u)-f_{u_xu_{xx}}]+u_{xtt}f_{u_xu_{xx}}$$

$$+u_{txx}[-f_{u_xu_{tx}}+f_{u_xu_{tx}u_{xx}}(u_{tt}-u_{xx}-u)]\}+u_{xt}\{-f_{xu_t}$$

$$+f_{xu_tu_{xx}}(u_{tt}-u_{xx}-u)+u_x[-f_{uu_t}+f_{uu_tu_{xx}}(u_{tt}-u_{xx}-u)-f_{u_tu_{xx}}]$$

$$+ u_{xx}[-f_{u_t u_{xx}} + f_{u_t u_{xx} u_{xx}}(u_{tt} - u_{xx} - u) - f_{u_t u_{xx}}] + u_{xtt} f_{u_t u_{xx}}$$

$$+ u_{txx}[-f_{u_t u_{tx}} + f_{u_{tx} u_{xx}}(u_{tt} - u_{xx} - u)]\} + u_{xxx}\{-2f_{x u_{xx}} + f_{x u_{xx} u_{xx}}(u_{tt} - u_{xx} - u)$$

$$+ u_x[-2f_{u u_{xx}} - f_{u_{xx} u_{xx}} + f_{u u_{xx} u_{xx}}(u_{tt} - u_{xx} - u)] + [-f_{u_x} + f_{u_x u_{xx}}(u_{tt} - u_{xx} - u)]$$

$$+ u_{xx}[-2f_{u_x u_{xx}} + f_{u_x u_{xx} u_{xx}}(u_{tt} - u_{xx} - u)] + u_{xt}[-2f_{u_t u_{xx}} + f_{u_t u_{xx} u_{xx}}$$

$$+ f_{u_t u_{xx} u_{xx}}(u_{tt} - u_{xx} - u)] + u_{xxx}[-3f_{u_{xx} u_{xx}} + f_{u_{xx} u_{xx} u_{xx}}(u_{tt} - u_{xx} - u)]$$

$$+ u_{xtt} f_{u_{xx} u_{xx}} + u_{txx}[-2f_{u_{xx} u_{tx}} + f_{u_{xx} u_{xx} u_{tx}}(u_{tt} - u_{xx} - u)]\}$$

$$+ u_{xtt}\{f_{x u_{xx}} + u_x f_{u u_{xx}} + u_{xx} f_{u_x u_{xx}} + u_{tx} f_{u_t u_{xx}} + u_{xxx} f_{u_{xx} u_{xx}} + u_{txx} f_{u_{tx} u_{tx}}\}$$

$$+ u_{txx}\{-f_{x u_{tx}} + f_{x u_{xx} u_{tx}}(u_{tt} - u_{xx} - u) + u_x[-f_{u u_{tx}} + f_{u u_{xx} u_{tx}}(u_{tt} - u_{xx} - u) - f_{u_{tx} u_{xx}}]$$

$$+ u_{xx}[-f_{u_x u_{tx}} + f_{u_x u_{xx} u_{tx}}(u_{tt} - u_{xx} - u)] + [-f_{u_t} + f_{u_t u_{xx}}(u_{tt} - u_{xx} - u)]$$

$$+ u_{tx}[-f_{u_t u_{tx}} + f_{u_t u_{xx} u_{tx}}(u_{tt} - u_{xx} - u)] + u_{ttx} f_{u_{xx} u_{tx}} + u_{xxx}[-2f_{u_{xx} u_{tx}}$$

$$+ f_{u_{tx} u_{xx} u_{xx}}(u_{tt} - u_{xx} - u)] + u_{txx}[-f_{u_{tx} u_{tx}} + f_{u_{tx} u_{tx} u_{xx}}(u_{tt} - u_{xx} - u)]\}$$

$$+ f_{tt} + u_t f_{tu} + u_{tt} f_{t u_t} + u_{tx} f_{t u_x} + u_{txx} f_{t u_{xx}} + u_{ttx} f_{t u_{tx}} + u_t\{f_{ut}$$

$$+ u_t f_{uu} + u_{tt} f_{u u_t} + u_{tx} f_{u u_x} + u_{txx} f_{u u_{xx}} + u_{ttx} f_{u u_{tx}}\}$$

$$+ u_{tt}\{f_{t u_t} + u_t f_{uu} + f_u + u_{tt} f_{u_t u_t} + u_{tx} f_{u_t u_x}$$

$$+ u_{txx} f_{u_t u_{xx}} + u_{ttx} f_{u_t u_{tx}}\} + u_{tx}\{f_{t u_x} + u_t f_{u u_x} + u_{tt} f_{u_t u_x}$$

$$+ u_{tx} f_{u_x u_x} + u_{txx} f_{u_x u_{xx}} + u_{ttx} f_{u_x u_{tx}}\} + u_{txx}\{f_{t u_{xx}} + u_t f_{u u_{xx}}$$

$$+ u_{tt} f_{u_t u_{xx}} + u_{tx} f_{u_x u_{xx}} + u_{txx} f_{u_{xx} u_{xx}} + u_{ttx} f_{u_{tx} u_{xx}}\}$$

$$+ u_{ttt} f_{u_t} + u_{ttx}\{f_{t u_{tx}} + u_t f_{u u_{tx}} + u_{tt} f_{u_t u_{tx}} + u_{tx} f_{u_x u_{tx}}$$

$$+ f_{u_x} + u_{txx} f_{u_{xx} u_{tx}} + u_{ttx} f_{u_{tx} u_{tx}}\} = 0 \quad \text{(A3)}$$

Comparing monomials (starting from second and higher derivatives) and then solving the resulting system of equations we obtain:

$$f = (c_1 t + c_2)u_x + (c_1 x + c_3)u_t + c_4 \exp(-c(t+x) - c_5 \exp(\frac{t-x}{4c}).$$

To find the conserved vector, we take for example, $c_2 \neq 0.$, thus f is $f(x,t,u,u_x,u_t,u_{xx},u_{xt}) = u_x$. Then we solve the following equation:

$$u_x(u_{tt} - u_{xx} - u) = D_t T + D_x S, \tag{A4}$$

where $T = T(x,t,u,u_x,u_t), \quad S = S(x,t,u,u_x,u_t)$.

Using the expression for total derivative into Eq. (A4) and comparing coefficients, we get:

$$u_x = \frac{\partial T}{\partial u_t} \tag{A5}$$

$$-u_x = \frac{\partial S}{\partial u_x}, \tag{A6}$$

$$\frac{\partial T}{\partial u_x} + \frac{\partial S}{\partial u_t} = 0, \tag{A7}$$

$$\frac{\partial T}{\partial t} + u_t \frac{\partial T}{\partial u} + \frac{\partial s}{\partial x} + u_x \frac{\partial s}{\partial u} = -u u_x \tag{A8}$$

From Eq. (A5), we get

$$T = u_x u_t + A(x,t,u,u_x) \tag{A9}$$

Using Eq. (A9) into Eq. (A7) yields

$$u_t + \frac{\partial A}{\partial u_x} + \frac{\partial S}{\partial u_t} = 0 \tag{A10}$$

From Eq. (A10), we infer that

$$S = \frac{-u_t^2}{2} - u_t \frac{\partial A}{\partial u_x} + B(x, t, u, u_x) \tag{A11}$$

Using Eq. (A11) into Eq. (A6),gives:

$$-u_x = -u_t \frac{\partial^2 A}{\partial u_x^2} + \frac{\partial B}{\partial u_x} \tag{A12}$$

Comparing the coefficients with previous equation gives:

$$u_t : 0 = -\frac{\partial^2 A}{\partial u_x^2}, \tag{A13}$$

$$1 : -u_x = \frac{\partial B(x, t, u, u_x)}{\partial u_x}. \tag{A14}$$

From Eqs. (A13) and (A14), we respectively obtain

$$A(x, t, u, u_x) = \alpha(x, t, u)u_x + \beta(x, t, u) \tag{A15}$$

and

$$B(x, t, u, u_x) = \frac{-1}{2} u_x^2 + \gamma(x, t, u) \tag{A16}$$

Therefore,

$$T = u_x u_t + \alpha(x,t,u)u_x + \beta(x,t,u) \tag{A17}$$

and

$$S = -\frac{1}{2}u_t^2 - \alpha(x,t,u)u_t - \frac{1}{2}u_x^2 + \gamma(x,t,u) \tag{A18}$$

Now using Eq. (A18) into Eq. (A8), we get:

$$-uu_x = \alpha_t u_x + \beta_t + u_t(\alpha_u u_x + \beta_u) + (-\alpha_x u_t + \gamma_x + u_x(-\alpha_u u_t + \gamma_u) \tag{A19}$$

Comparing coefficients of u_t, u_x in Eq. (A19), yields:

$$u_x : -u = \alpha_t + \gamma_u, \tag{A20}$$

$$u_t : 0 = \beta_u - \alpha_x, \tag{A21}$$

$$1 : 0 = \beta_t + \gamma_x. \tag{A22}$$

If $\beta_u, \alpha_u = 0$, then $\alpha = \alpha(x,t)$ and $\beta = \beta(x,t)$. Thus from Eq. (A20), we infer that:

$$\gamma = \frac{-1}{2}u^2 - \alpha_t u + a(x,t) \tag{A23}$$

Since $\beta_u = 0$, we conclude from Eq. (A22) that $\alpha = \alpha(t)$.

Thus, $\gamma_x = a_x$ and Eq. (A20), give rise to the following condition:

$$a_x + \beta_t = 0 \tag{A24}$$

Consequently, the forms of T and S are

$$
\begin{aligned}
T &= u_x u_t + \alpha(t) u_x + \beta(x,t) \\
S &= -\frac{1}{2}u_t^2 - \alpha(t)u_t - \frac{1}{2}u_x^2 - \frac{1}{2}u^2 - \alpha(t)u + a(x,t)
\end{aligned}
\tag{A25}
$$

Appendix B

Reduction- case I.a

Algebra	Reduction	z	v
$[X_1, X_2] = 0$	$v_z + N^2cv = 0$	t	u
$[X_1, X_3] = 0$	$v_{zz} - \frac{1}{v}v_z^2 + \frac{N^2c\beta}{\gamma}v = 0$	$x\cos y$	$(\beta - u)\exp(-N^2ct)$
$[X_1, X_4] = 0$	$\frac{\gamma}{\beta-v}v_{zz} + \frac{\gamma}{(\beta-v)^2}v_z^2 - N^2cv = 0$	$x\cos y$	u
$[X_2, X_3] = 0$	$v_{zz} - \frac{1}{v}v_z^2 + \frac{N^2c\beta}{\gamma}v = 0$	$x\sin y$	$(\beta - u)\exp(-N^2ct)$
$[X_2, X_4] = 0$	$\frac{\gamma}{\beta-v}v_{zz} + \frac{\gamma}{(\beta-v)^2}v_z^2 - N^2cv = 0$	$x\sin y$	u
$[X_3, X_5] = 0$	$z^2v_{zz} - \frac{z^2}{v}v_z^2 + zv_z + \frac{N^2c\beta}{\gamma}z^2v = 0$	x	$(\beta - u)\exp(N^2ct)$
$[X_4, X_5] = 0$	$\frac{\gamma}{\beta-v}z^2v_{zz} + \frac{\gamma}{(\beta-v)^2}z^2v_z^2 + \frac{\gamma}{\beta-v}zv_z - N^2cvz^2 = 0$	x	u

Reduction- case I.b

Algebra	Reduction	z	v
$[X_1, X_2] = 0$	$v_z + N^2 cv = 0$	t	u
$[X_1, X_3] = X_1$	$v_z - 2\gamma\alpha^{\frac{1+\alpha}{\alpha}}(2\alpha+1)v^{\frac{1}{\alpha}+1} + N^2 cv = 0$	t	$(x\cos y)^{2\alpha}u$
$[X_1, X_4] = 0$	$v_{zz} + \frac{1}{\alpha}\frac{1}{v}v_z^2 = 0$	$x\cos y$	$\exp(N^2 ct)u$
$[X_1, X_5] = 0$	$\gamma(\alpha v)^{\frac{1}{\alpha}}v_{zz} + \gamma(\alpha v)^{\frac{1}{\alpha}-1}v_z^2 - N^2 cv = 0$	$x\cos y$	u
$[X_2, X_3] = X_2$	$2\gamma\alpha^{\frac{1}{\alpha}+1}(2\alpha+1)v^{\frac{1}{\alpha}+1} - N^2 cv - v_z = 0$	t	$(x\sin y)^{-2\alpha}u$
$[X_2, X_4] = 0$	$\gamma\alpha^{\frac{1}{\alpha}}v^{\frac{1}{\alpha}}v_{zz} + \gamma\alpha^{\frac{1}{\alpha}}v^{\frac{1}{\alpha}-1}v_z^2 = 0$	$x\sin y$	$\exp(N^2 ct)u$
$[X_3, X_4] = 0$	$v_{zz} + \frac{1}{\alpha}v_z^2 + 4\alpha(\alpha+1)v^{\frac{1}{\alpha}+1} = 0$	y	$e^{N^2 ct}x^{-2\alpha}u$
$[X_3, X_5] = 0$	$\gamma\alpha^{\frac{1}{\alpha}}v^{\frac{1}{\alpha}}v_{zz} + \gamma\alpha^{\frac{1}{\alpha}-1}v^{\frac{1}{\alpha}-1}v_z^2 + 4\gamma\alpha^{\frac{1}{\alpha}+1}(\alpha+1)v^{\frac{1}{\alpha}+1} = 0$	y	$x^{-2\alpha}u$
$[X_3, X_6] = 0$	$v_z - 4\gamma\alpha^{\frac{1}{\alpha}+1}v^{\frac{1}{\alpha}+1} + N^2 cv = 0$	t	$x^{-2\alpha}u$
$[X_5, X_6] = 0$	$\gamma z^2(\alpha v)^{\frac{1}{\alpha}}v_{zz} + \gamma z^2(\alpha v)^{\frac{1}{\alpha}-1}v_z^2 - N^2 cz^2 v = 0$	x	u

Reduction- case II.b

Algebra	Reduction	z	v
$[X_1, X_2] = -2X_2$	$k(v)v_{zz} + k_v v_z^2 - N^2 cv = 0$	y	u
$[X_1, X_3] = 0$	$z^2 k(v)v_{zz} + z^2 k_v v_z^2 + zk(v)v_z + \frac{1}{2}z^3 v_z - N^2 cv = 0$	$xt^{-\frac{1}{2}}$	u
$[X_2, X_3] = 0$	$z^2 k(v)v_{zz} + z^2 k_v v_z^2 + zk(v)v_z - N^2 cv = 0$	x	u

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3. *Awards and Scholarships*

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KFUPM, Dhahran, KSA	2009 – 2013	Lecturer B
KFUPM, Dhahran, KSA	2004 – 2009	Research Assistant
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- **King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia,** Calculus I, Calculus II.
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6. *Seminars*

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